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SMALL-ANGLE STABILITY ANALYSIS  
OF A LINEAR CONTROL SYSTEM FOR  
A HIGH-POWER COMMUNICATION SATELLITE

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16. Abstract <p>A small-angle stability analysis is presented for one particular configuration of a high-power communication satellite having a linear control system. Both the central body and the solar array are treated as rigid bodies. The control system studied consists of three-axis control of the central body and one-axis control of the solar-array rotation relative to the central body. The results of the report yield preliminary indications of the relation of stability to satellite inertias and control gains.</p>					
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# SMALL-ANGLE STABILITY ANALYSIS OF A LINEAR CONTROL SYSTEM FOR A HIGH-POWER COMMUNICATION SATELLITE

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## SUMMARY

This report presents a small-angle stability analysis for one particular configuration of a high-power communication satellite having a linear control system. Both the central body and the solar array are treated as rigid bodies. The configuration studied is one for which the solar array has one degree of freedom with respect to the central body, and the antennas are rigidly attached to the central body. The control system studied consists of three-axis control of the central body and one-axis control of the solar-array rotation relative to the central body. The results of the report yield preliminary indications of the relation of stability to satellite inertias and control gains.

Assumptions are made which allow the equations of motion to be linearized and which make the resulting stability analysis tractable. The eighth-order characteristic equation is separated into two second-order equations for the roll and yaw axes of the spacecraft and a fourth-order equation for the pitch axis. For positive control gains, stability is achieved for the roll and yaw axes. Not all positive values of control gains will yield stability for the pitch axis. Pitch-axis stability is achieved only when the control gains are sufficiently large.

When the central body and solar array do not have coincident centers of mass, steady-state pointing errors result. For reasonable values of inertias and control gains, the steady-state pointing errors are negligible. The degree of stability is not significantly changed when the mass centers are not coincident.

## INTRODUCTION

The next generation of high-power communication satellites will use large sun-oriented solar arrays as a primary power source. Typical configurations of such communication satellites have one or more antennas mounted on a relatively dense central

body. The central body must be tightly controlled in order to transmit narrow radio-frequency beams from synchronous equatorial orbit to high-gain receiver antennas on the ground. The dual control requirements of earth-pointing the central body and sun-pointing the solar array require one or perhaps two degrees of freedom between the central body and the solar array.

To analyze the attitude control of such communication satellites, the equations of motion of multibody spacecraft have to be derived. General methods of deriving these nonlinear equations have appeared in a number of reports (refs. 1 to 3) in recent years. The process of determining the stability of a control system often involves using standard techniques, such as root locus, which can be applied only to a system of linear equations. This report presents the assumptions and analysis required to reduce the nonlinear dynamic equations to a form in which the standard linear stability techniques can be easily applied.

The dynamic equations used in this report are for a spacecraft consisting of two rigid bodies - a central body and a solar array. The configuration studied is one for which the solar array has one degree of freedom with respect to the central body, and the spacecraft antennas are rigidly attached to the central body. The control system studied consists of three-axis control of the central body and one-axis control of the solar-array rotation relative to the central body.

The method of attack is to derive the generalized equations of motion. The full derivation is given in appendix A. The generalized equations are then linearized in appendix B. Symbols are defined in appendix C. The characteristic equation of the system is obtained from the linearized equations of motion. Approximations are made to separate the eighth-order characteristic equation into two second-order equations for the roll and yaw axes of the spacecraft and a fourth-order equation for the pitch axis. Stability is analyzed first for the case of the central body and solar array having coincident centers of mass. The analysis is then generalized for distinct centers of mass.

## COORDINATE SYSTEMS

Before deriving equations of motion, it is necessary to define coordinate systems suitable to the satellite configuration. The configuration to be studied in this report, shown in figure 1, is the one proposed for the Canadian Technology Satellite. The same configuration is also being considered for the ATS-H and ATS-I satellites.

These satellites have a dense central body with rigidly attached antennas and a large solar array. The attitude control system maintains the central body in an earth-oriented position in order to satisfy antenna-pointing requirements. The large solar array is sun-oriented. To maintain earth-pointing and sun-pointing requirements, the solar array rotates with respect to the central body at a nominal angular velocity of one revolu-

tion per day. Since there is only one degree of rotational freedom of the solar array with respect to the central body, the nominal sun-pointing error is zero only during the vernal and autumnal equinoxes, when the sun is in the equatorial plane. The nominal sun-pointing error is a maximum ( $23.5^\circ$ ) during the solstices.

The central-body coordinate system and the orbital reference coordinate system are shown in figure 2. The orbital reference coordinate system, specified by unit vectors  $\bar{x}_r$ ,  $\bar{y}_r$ ,  $\bar{z}_r$  along the coordinate axes, has its origin at the center of mass of the central body. The positive  $\bar{z}_r$ -axis points to the center of the earth, the positive  $\bar{x}_r$ -axis lies along the velocity vector of the assumed circular orbit, and the  $\bar{y}_r$ -axis completes the right-handed set. The central-body coordinate system, specified by unit vectors  $\bar{x}_1$ ,  $\bar{y}_1$ ,  $\bar{z}_1$  along the coordinate axes, is fixed in the central body and has its origin at the central-body center of mass. The  $\bar{x}_1$ ,  $\bar{y}_1$ , and  $\bar{z}_1$  axes are oriented such that they coincide with  $\bar{x}_r$ ,  $\bar{y}_r$ , and  $\bar{z}_r$  when the central body is in its nominal orientation. The  $\bar{x}_1$ ,  $\bar{y}_1$ , and  $\bar{z}_1$  axes will be called the roll, pitch, and yaw axes of the central body. The central-body coordinate system is obtained from the orbital reference coordinate system by three rotations: (1) rotation about  $\bar{x}_r$ -axis through an angle  $\theta_1$ ; (2) rotation about the  $\bar{y}_r$ -axis through an angle  $\theta_2$ ; (3) rotation about the  $\bar{z}_r$ -axis through an angle  $\theta_3$ . The objective of the central-body control system is to null out the pointing errors  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$ .

The solar-array coordinate system, shown in figure 3, is fixed in the solar array and is specified by unit vectors  $\bar{x}_2$ ,  $\bar{y}_2$ , and  $\bar{z}_2$  along the coordinate axes. The origin of the solar-array coordinate system is at the solar-array center of mass. The positive  $\bar{y}_2$ -axis is along the solar-array shaft and is parallel to the positive  $\bar{y}_1$ -axis. The  $\bar{z}_2$ -axis is perpendicular to the solar-array surface, and the  $\bar{x}_2$ -axis completes the right-handed set. The  $\bar{x}_2$ ,  $\bar{y}_2$ , and  $\bar{z}_2$  axes will be called the roll, pitch, and yaw axes of the solar array. Let  $\bar{x}'_1$ ,  $\bar{y}'_1$ , and  $\bar{z}'_1$  be unit vectors along the axes of a coordinate system whose axes are parallel to the central-body axes but whose origin is at the solar-array center of mass. The angle  $\varphi$  shown in figure 3 is the angle between  $\bar{x}'_1$  and  $\bar{x}_2$ , or equivalently the angle between  $\bar{z}'_1$  and  $\bar{z}_2$ .

Let  $\bar{\delta}$  be a unit vector originating from the solar-array center of mass and pointing to the sun, as shown in figure 4. Then the true solar-array pointing error is the angle between  $\bar{z}_2$  and  $\bar{\delta}$ . This pointing error, however, may have a nominal value as large as  $23.5^\circ$ . For attitude control purposes, we assume that the error sensed by the control system is the angle  $\sigma$  shown in figure 4. The vector  $\bar{\delta}_p$  is a unit vector obtained from the projection of  $\bar{\delta}$  into the  $\bar{x}_2 - \bar{z}_2$  plane. The angle subtended by  $\bar{z}_2$  and  $\bar{\delta}_p$  is  $\sigma$ .

The four rotational degrees of freedom of the system are specified by the angles  $\theta_1$ ,  $\theta_2$ ,  $\theta_3$ , and  $\varphi$ . The angles  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$  will be small angles; whereas, the angle  $\varphi$  nominally increases at the rate of  $2\pi$  radians per day. Since the four angular coordinates used for the stability analysis must remain small in order to justify linear-



izing the equations of motion,  $\varphi$  cannot be used as one of the coordinates. The fourth coordinate must be defined as a perturbation from a nominal motion.

The geometry for defining a suitable fourth coordinate is shown in figure 5. The plane of the figure is the equatorial plane, and the line  $L$  is the projection of the earth-sun line into the equatorial plane. The line  $L$  may be considered fixed for the time span considered. Let  $\bar{x}'_r(t)$  and  $\bar{z}'_r(t)$  be unit vectors, at time  $t$ , along the axes of a coordinate system whose axes are parallel to the orbital reference axes but whose origin is at the center of the earth. The angle subtended by the line  $L$  and the vector  $\bar{z}'_r(0)$  is  $\alpha$ . The angle subtended by  $\bar{x}'_r(0)$  and  $\bar{x}'_r(t)$  is  $\omega_0 t$ , where  $\omega_0$  is the angular velocity of the synchronous orbit.

The fourth coordinate  $\theta_4$  is defined as

$$\theta_4(t) = \varphi(t) - (\alpha + \omega_0 t) \quad (1)$$

If the spacecraft is in its nominal position at time  $t$ , the line  $L$  is coincident with  $\bar{z}_2(t)$ , and  $\bar{z}_r(t)$  is coincident with  $\bar{z}_1(t)$ . Thus,  $\varphi(t)$ , the angle between  $\bar{z}_2(t)$  and  $\bar{z}_1(t)$ , will be the angle between  $L$  and  $\bar{z}'_r(t)$ , which by figure 5 is the angle  $\alpha + \omega_0 t$ . From equation (1), we conclude that  $\theta_4(t)$  is zero when the spacecraft is in its nominal position. Thus,  $\theta_4$  is a small angle and is a suitable fourth coordinate.

When the sun is in the equatorial plane, the solar-array error  $\sigma$  is the sum of  $\theta_2$  and  $\theta_4$ . When the sun is not in the equatorial plane, the central-body roll and yaw errors  $\theta_1$  and  $\theta_3$  will also contribute to  $\sigma$ . In the general case,  $\sigma$  is given by (see appendix B)

$$\sigma = (G \sin \varphi) \theta_1 + \theta_2 + (G \cos \varphi) \theta_3 + \theta_4 \quad (2)$$

where  $G$  is defined as

$$G = \frac{-\sin \theta_s \sin i}{\sqrt{1 - \sin^2 \theta_s \sin^2 i}} \quad (3)$$

The angle  $i$  of equation (3) is the inclination of the equatorial plane to the ecliptic plane, and the angle  $\theta_s$  is the apparent angular displacement of the sun from the autumnal equinox measured in the ecliptic plane, as shown in figure 6.

## ASSUMPTIONS

It is necessary to make assumptions which will justify linearizing the equations of motion and which will make the resulting stability analysis tractable. The assumptions

are

(1) The error signals  $\theta_i$  and  $\dot{\theta}_i$ ,  $i = 1, 2, 3$ , are available from sensor data and can be used for controlling the central body. Letting  $T_{c1}$ ,  $T_{c2}$ ,  $T_{c3}$  be the central-body control torques about the  $\bar{x}_1$ ,  $\bar{y}_1$ , and  $\bar{z}_1$  axes of the central body, it is assumed that

$$T_{ci} = -K_i \theta_i - C_i \dot{\theta}_i \quad i = 1, 2, 3 \quad (4)$$

where the  $K_i$  and  $C_i$  are specified gain constants.

(2) For controlling the solar array, it is assumed that a sun sensor is used for sensing  $\sigma$  and that a tachometer or some other rate sensor is used for sensing the angular velocity  $\dot{\phi}$  of the solar array with respect to the central body. A motor torque is used for rotating the solar array relative to the central body. The motor torque  $T_m$  is given by

$$T_m = K_4 \sigma + C_4 (\dot{\phi} - \omega_0) \quad (5)$$

where  $K_4$  and  $C_4$  are specified gain constants. It is assumed that  $T_m$  is the torque on the central body.

(3) The satellite remains in a circular synchronous equatorial orbit.

(4) The angles  $\theta_1$ ,  $\theta_2$ ,  $\theta_3$ , and  $\theta_4$  and angular rates  $\dot{\theta}_1$ ,  $\dot{\theta}_2$ ,  $\dot{\theta}_3$ , and  $\dot{\theta}_4$  are small.

(5) The principal axes of inertia of the central body coincide with the central-body coordinate axes, and the principal axes of inertia of the solar array coincide with the solar-array coordinate axes. The central-body inertias about the  $x$ ,  $y$ , and  $z$  axes of the central body are denoted by  $I_{1x}$ ,  $I_{1y}$ , and  $I_{1z}$ ; and the solar-array inertias about the  $x$ ,  $y$ , and  $z$  axes of the solar array are denoted by  $I_{2x}$ ,  $I_{2y}$ , and  $I_{2z}$ . These inertias are assumed constant.

(6) The solar-array center of mass lies along the solar-array shaft.

(7) The dominant torques on the spacecraft are the control torques given in equations (4) and (5) and the constraint torque which constrains the solar array to have only one rotational degree of freedom relative to the central body. All other torques are neglected.

(8) Nonlinear terms in the equations of motion are neglected.

## EQUATIONS OF MOTION

The rotational equations of motion are derived in appendix A. The method of equation derivation is taken from reference 1. In appendix B, the assumptions of the pre-

vious section are used to derive linearized equations of motion. Let  $\bar{u}$  be the vector from the central-body center of mass to the solar-array center of mass, and let  $u_{1x}$ ,  $u_{1y}$ ,  $u_{1z}$  be the components of  $\bar{u}$  in the central-body coordinate system. The linearized equations of motion can be written in matrix form as

$$[M]\{\ddot{\theta}\} + [C]\{\dot{\theta}\} + [K]\{\theta\} = \{f\} \quad (6)$$

where  $[M]$ ,  $[C]$ ,  $[K]$ , and  $\{f\}$  are given by

$$[M] = \begin{bmatrix} I_{1x} + I_{2x} + (I_{2z} - I_{2x})\sin^2\varphi & 0 & (I_{2z} - I_{2x})\sin\varphi \cos\varphi & 0 \\ 0 & I_{1y} & 0 & 0 \\ (I_{2z} - I_{2x})\sin\varphi \cos\varphi & 0 & I_{1z} + I_{2z} - (I_{2z} - I_{2x})\sin^2\varphi & 0 \\ 0 & I_{2y} & 0 & I_{2y} \end{bmatrix} + \frac{M_1 M_2}{M_T} \begin{bmatrix} u_{1y}^2 + u_{1z}^2 & -u_{1x}u_{1y} & -u_{1x}u_{1z} & 0 \\ -u_{1y}u_{1x} & u_{1x}^2 + u_{1y}^2 & -u_{1y}u_{1z} & 0 \\ -u_{1z}u_{1x} & -u_{1z}u_{1y} & u_{1x}^2 + u_{1y}^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (7)$$

$$[C] = \begin{bmatrix} C_1 + \omega_o[2(I_{2z} - I_{2x})\sin\varphi \cos\varphi] & 0 & -\omega_o[I_{1x} - I_{1y} + I_{1z} + 2I_{2x} + 2(I_{2z} - I_{2x})\sin^2\varphi] & 0 \\ 0 & C_2 & 0 & -C_4 \\ \omega_o[I_{1x} - I_{1y} + I_{1z} + 2I_{2x} + 2(I_{2z} - I_{2x})\cos^2\varphi] & 0 & C_3 - \omega_o[2(I_{2z} - I_{2x})\sin\varphi \cos\varphi] & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix} + \frac{\omega_o M_1 M_2}{M_T} \begin{bmatrix} 0 & 2u_{1y}u_{1z} & -2u_{1y}^2 & 0 \\ -2u_{1y}u_{1z} & 0 & 2u_{1x}u_{1y} & 0 \\ 2u_{1y}^2 & -2u_{1x}u_{1y} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (8)$$



$$\begin{aligned}
[K] = & \begin{bmatrix} K_1 - \omega_o^2 [I_{1z} - I_{1x} + I_{2x} + (I_{2z} - I_{2x}) \sin^2 \varphi] & 0 & -\omega_o^2 [(I_{2z} - I_{2x}) \sin \varphi \cos \varphi] & 0 \\ -K_4 G \sin \varphi & K_2 - K_4 & -K_4 G \cos \varphi & -K_4 \\ -\omega_o^2 [(I_{2z} - I_{2x}) \sin \varphi \cos \varphi] & 0 & K_3 - \omega_o^2 [I_{1x} - I_{1y} + I_{2x} + (I_{2z} - I_{2x}) \cos^2 \varphi] & 0 \\ K_4 G \sin \varphi & K_4 & K_4 G \cos \varphi & K_4 \end{bmatrix} \\
& + \frac{\omega_o^2 M_1 M_2}{M_T} \begin{bmatrix} 4(u_{1z}^2 - u_{1y}^2) & 3u_{1x}u_{1y} & -u_{1x}u_{1z} & 0 \\ 4u_{1x}u_{1y} & 3(u_{1z}^2 - u_{1x}^2) & u_{1y}u_{1z} & 0 \\ -4u_{1z}u_{1x} & -3u_{1z}u_{1y} & u_{1x}^2 - u_{1y}^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (9)
\end{aligned}$$

$$\{f\} = \frac{\omega_o^2 M_1 M_2}{M_T} \begin{Bmatrix} 4u_{1y}u_{1z} \\ -3u_{1x}u_{1z} \\ u_{1x}u_{1y} \\ 0 \end{Bmatrix} \quad (10)$$

If the matrices  $[M]$ ,  $[C]$ , and  $[K]$  have constant elements, the stability characteristics of the system are found by taking the Laplace transform of matrix equation (6), assuming zero initial conditions

$$s^2[M]\{\bar{\theta}\} + s[C]\{\bar{\theta}\} + [K]\{\bar{\theta}\} = \{f\} \quad (11)$$

which can be written as

$$[Z(s)]\{\bar{\theta}\} = \{f\} \quad (12)$$

where  $[Z(s)]$  is given by

$$[Z(s)] = s^2[M] + s[C] + [K] \quad (13)$$

We can define a matrix of transfer functions given by

$$[Z(s)]^{-1} = \frac{[A(s)]}{\det[Z(s)]} \quad (14)$$

where

$$[A(s)] = \text{adj}[Z(s)] \quad (15)$$

The transformed solution  $\{\bar{\theta}\}$  is

$$\{\bar{\theta}\} = [Z(s)]^{-1} \{\bar{f}\} = \frac{[A(s)]\{\bar{f}\}}{\det[Z(s)]} \quad (16)$$

The solution is obtained by writing the inverse Laplace transformation,

$$\{\theta\} = \mathcal{L}^{-1}\{\bar{\theta}\} = \mathcal{L}^{-1}\left(\frac{[A(s)]\{\bar{f}\}}{\det[Z(s)]}\right) \quad (17)$$

The poles are given by those values of  $s$  which satisfy

$$\det[Z(s)] = 0 \quad (18)$$

Equation (18) is called the characteristic equation of the system and the values of  $s$  satisfying the characteristic equation are called the characteristic roots. If the real parts of all characteristic roots are negative, the system is stable.

If the matrices  $[M]$ ,  $[C]$ , and  $[K]$  are not constant, evaluating the characteristic roots by equation (18) is not a strictly valid procedure for determining stability. It is customary, however, to evaluate  $[M]$ ,  $[C]$ , and  $[K]$  at a fixed time and to regard the characteristic roots of equation (18) as an indication of "local" stability at the fixed time (ref. 4).

Precise conditions for which the "local" stability procedure is valid are not known. It is felt, however, that for slowly time-varying matrices an indication of stability is achieved.

Only those elements of  $[M]$ ,  $[C]$ , and  $[K]$  containing  $G$ ,  $\sin \varphi$ , and  $\cos \varphi$  are time dependent. The parameter  $G$  has a frequency of 1 cycle per year. Terms containing  $\sin \varphi$  and  $\cos \varphi$  have a frequency of 1 cycle per day because  $\dot{\varphi}$  is nominally equal to  $\omega_0$ .

## STABILITY FOR $\bar{u} = \bar{0}$

We assume in this section that  $\bar{u} = \bar{0}$ , or that the centers of mass of the central body and solar array coincide. In equations (7), (8), and (9),  $[M]$ ,  $[C]$ , and  $[K]$  are each expressed as the sum of two matrices, where one matrix is independent of  $\bar{u}$  and the other

is dependent on  $\bar{u}$ . For  $\bar{u} = \bar{0}$ , the second matrices on the right-hand side of equations (7), (8), and (9) will be zero.

Letting  $z_{ij}$  be the element in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of  $[Z]$ , equations (7), (8), (9), and (13) imply that  $z_{12}$ ,  $z_{14}$ ,  $z_{32}$ , and  $z_{34}$  are all zero. The characteristic equation can be written as

$$\det[Z] = (z_{11}z_{33} - z_{13}z_{31})(z_{22}z_{44} - z_{24}z_{42}) = 0 \quad (19)$$

The characteristic polynomial contains the term  $z_{11}z_{22}z_{33}z_{44}$ ; and each  $z_{ii}$ ,  $i = 1, 2, 3$ , and  $4$ , is a second-degree polynomial in  $s$ . Thus, the characteristic equation is an eighth-degree polynomial set equal to zero. The characteristic equation is independent of  $z_{21}$ ,  $z_{23}$ ,  $z_{41}$ , and  $z_{43}$ . These four terms are the only ones which contain the parameter  $G$ . Thus, the stability of the system is independent of the degree of coupling of the central-body roll and yaw errors  $\theta_1$  and  $\theta_3$  into the solar-array pointing error  $\sigma$ .

The factorization of equation (19) implies the separability of the characteristic equation into two equations:

$$z_{11}z_{33} - z_{13}z_{31} = 0 \quad (20)$$

$$z_{22}z_{44} - z_{24}z_{42} = 0 \quad (21)$$

Equation (20) contains only roll- and yaw-axis parameters and will be referred to as the roll-yaw characteristic equation. Equation (21) contains only pitch-axis parameters and will be referred to as the pitch characteristic equation.

### Roll-Yaw Stability

The roll-yaw characteristic equation is dependent on  $z_{11}$ ,  $z_{33}$ ,  $z_{13}$ ,  $z_{31}$ . These four terms are

$$\begin{aligned} z_{11} = & \left[ (I_{1x} + I_{2x}) + (I_{2z} - I_{2x})\sin^2\varphi \right] s^2 + \left[ C_1 + 2\omega_o(I_{2z} - I_{2x})\sin\varphi \cos\varphi \right] s \\ & + K_1 - \omega_o^2 \left[ I_{1z} - I_{2y} + I_{2x} + (I_{2z} - I_{2x})\sin^2\varphi \right] \end{aligned} \quad (22)$$

$$z_{33} = \left[ (I_{1z} + I_{2z}) - (I_{2z} - I_{2x}) \sin^2 \varphi \right] s^2 + \left[ C_3 - 2\omega_0 (I_{2z} - I_{2x}) \sin \varphi \cos \varphi \right] s + K_3 - \omega_0^2 \left[ I_{1x} - I_{1y} + I_{2x} + (I_{2z} - I_{2x}) \cos^2 \varphi \right] \quad (23)$$

$$z_{13} = \left[ (I_{2z} - I_{2x}) \sin \varphi \cos \varphi \right] s^2 - \omega_0 \left[ I_{1z} - I_{1x} + 2I_{2x} + 2(I_{2z} - I_{2x}) \sin^2 \varphi \right] s - \omega_0^2 \left[ (I_{2z} - I_{2x}) \sin \varphi \cos \varphi \right] \quad (24)$$

$$z_{31} = \left[ (I_{2z} - I_{2x}) \sin \varphi \cos \varphi \right] s^2 - \omega_0 \left[ I_{1y} - I_{1x} - I_{1z} - 2I_{2x} - 2(I_{2z} - I_{2x}) \cos^2 \varphi \right] s - \omega_0^2 \left[ (I_{2z} - I_{2x}) \sin \varphi \cos \varphi \right] \quad (25)$$

These four equations can be simplified by using several assumptions. First, assume that the inertia difference  $I_{2z} - I_{2x}$  is much smaller than the inertia sums  $I_{1x} + I_{2x}$  and  $I_{1z} + I_{2z}$ . This assumption is satisfied for the satellite configuration shown in figure 1. Second, assume that the gain constants  $C_1$ ,  $C_3$ ,  $K_1$ , and  $K_3$  satisfy the following inequalities:

$$C_i \gg 2\omega_0 I_{2x} \quad i = 1, 3$$

$$K_i \gg \omega_0^2 I_{2x} \quad i = 1, 3$$

Then equations (22) to (25) can be approximated by

$$z_{11} = (I_{1x} + I_{2x}) s^2 + C_1 s + K_1 \quad (26)$$

$$z_{33} = (I_{1z} + I_{2z}) s^2 + C_3 s + K_3 \quad (27)$$

$$z_{13} = 0 \quad (28)$$

$$z_{31} = 0 \quad (29)$$

The roll-yaw characteristic equation is then approximated by

$$\left[ (I_{1x} + I_{2x}) s^2 + C_1 s + K_1 \right] \left[ (I_{1z} + I_{2z}) s^2 + C_3 s + K_3 \right] = 0 \quad (30)$$

The first quadratic factor can be thought of as the roll characteristic equation

$$(I_{1x} + I_{2x})s^2 + C_1s + K_1 = 0 \quad (31)$$

because only roll-axis parameters appear. Similarly, the second quadratic factor can be thought of as the yaw characteristic equation

$$(I_{1z} + I_{2z})s^2 + C_3s + K_3 = 0 \quad (32)$$

Let  $s_{r1}$  and  $s_{r2}$  be the roots of the roll characteristic equation (31), and let  $s_{y1}$  and  $s_{y2}$  be the roots of the yaw characteristic equation (32). Assuming that

$$C_1^2 - 4K_1(I_{1x} + I_{2x}) < 0$$

$$C_3^2 - 4K_3(I_{1z} + I_{2z}) < 0$$

then the roll and yaw roots are complex conjugate pairs.

$$s_{r1}, s_{r2} = \frac{-C_1}{2(I_{1x} + I_{2x})} \pm j \frac{\sqrt{4K_1(I_{1x} + I_{2x}) - C_1^2}}{2(I_{1x} + I_{2x})} \quad (33)$$

$$s_{y1}, s_{y2} = \frac{-C_3}{2(I_{1z} + I_{2z})} \pm j \frac{\sqrt{4K_3(I_{1z} + I_{2z}) - C_3^2}}{2(I_{1z} + I_{2z})} \quad (34)$$

The stability for the roll and yaw axes is determined by the real parts of the roots.

$$\text{Re}(s_{r1}) = \text{Re}(s_{r2}) = \frac{-C_1}{2(I_{1x} + I_{2x})} \quad (35)$$

$$\text{Re}(s_{y1}) = \text{Re}(s_{y2}) = \frac{-C_3}{2(I_{1z} + I_{2z})} \quad (36)$$

Thus, for positive  $C_1$  and  $C_3$ , the approximate analysis results in stability for the roll and yaw axes. If either  $C_1$  or  $C_3$  is zero, there will be roots with zero real parts.

Thus, proportional feedback alone is not sufficient to achieve stability.

A locus of the roots of equations (33) and (34) is given in figure 7. The locus parameter is  $K_1/\omega_0^2(I_{1x} + I_{2x})$ , and the control gains and inertias are specified by

$$\frac{C_i}{K_i} = 1 \text{ sec} \quad i = 1, 3 \quad (37)$$

$$\frac{K_1}{K_3} = 1 \quad (38)$$

$$\frac{I_{1x} + I_{2x}}{I_{1z} + I_{2z}} = 1 \quad (39)$$

The gain ratios and inertia ratios specified by equations (37) to (39) are chosen to represent a realistic dynamic response of a high-power communication satellite configured as in figure 1. From equations (37) to (39), the roll roots coincide with the yaw roots in figure 7. If equation (37) is replaced by

$$\frac{C_i}{K_i} = 10 \text{ sec} \quad i = 1, 3$$

then the roll-yaw root locus is given by figure 8. With the greater rate feedback, the roots are further away from the imaginary axis.

### Pitch Stability

The pitch characteristic equation is dependent on  $z_{22}$ ,  $z_{44}$ ,  $z_{24}$ , and  $z_{42}$ . These four terms are

$$z_{22} = I_{1y}s^2 + C_2s + (K_2 - K_4) \quad (40)$$

$$z_{44} = I_{2y}s^2 + C_4s + K_4 \quad (41)$$

$$z_{24} = -C_4s - K_4 \quad (42)$$

$$z_{42} = I_{2y}s^2 + K_4 \quad (43)$$



The pitch characteristic equation (21) can be expanded to yield

$$(I_{1y}I_{2y})s^4 + (I_{2y}C_2 + I_{1y}C_4 + I_{2y}C_4)s^3 + (I_{1y}K_4 + C_2C_4 + I_{2y}K_2)s^2 + (C_2K_4 + C_4K_2)s + K_2K_4 = 0 \quad (44)$$

Unlike the roll-yaw characteristic equation, the pitch characteristic equation cannot be well approximated by two quadratic factors with  $K_2$  and  $C_2$  appearing in one quadratic factor and  $K_4$  and  $C_4$  appearing in the other.

One particular case worthy of investigation is the case  $C_4 = 0$ . Physically,  $C_4$  being zero means that no tachometer or other rate sensor is used to measure the body 2 rate error  $\dot{\theta}_4$ . When  $C_4$  is zero, equation (44) can be expressed as the product of two quadratic factors:

$$(I_{2y}s^2 + K_4)(I_{1y}s^2 + C_2s + K_2) = 0 \quad (45)$$

The characteristic roots  $\pm j\sqrt{K_4/I_{2y}}$ , of the first quadratic factor have zero real parts. Thus, stability is not achievable for  $C_4 = 0$ , and a rate sensor is required to sense  $\dot{\theta}_4$ .

A locus of the roots of the pitch characteristic equation (44) is given in figure 9. The locus parameter is  $K_4/\omega_0^2 I_{2y}$ , and the control gains and inertias are specified by

$$\frac{C_i}{K_i} = 1 \text{ sec} \quad i = 2, 4 \quad (46)$$

$$\frac{K_2}{K_4} = \frac{1}{4} \quad (47)$$

$$\frac{I_{2y}}{I_{1y}} = \frac{1}{4} \quad (48)$$

The gain ratios and inertia ratios specified by equations (46) to (48) are chosen to represent a realistic dynamic response of a high-power communication satellite configured as in figure 1.

Figure 9 shows that one pair of roots has a positive real part if  $K_4/\omega_0^2 I_{2y}$  is less than about  $2 \times 10^6$ . Unlike the roll-yaw case, not all positive values of  $K_2$ ,  $K_4$ ,  $C_2$ , and  $C_4$  yield pitch-axis stability. If equation (46) is replaced by

$$\frac{C_i}{K_i} = 10 \text{ sec} \quad i = 2, 4$$

then the locus of the pitch roots is given by figure 10. With the greater rate feedback, figure 10 shows that stability is achieved for all positive values of  $K_2$ ,  $K_4$ ,  $C_2$ , and  $C_4$ .

The coefficient of  $s^3$  in the pitch characteristic equation (44) is the sum of three terms. If the third term,  $I_{2y}C_4$ , is dropped, the pitch characteristic equation can be expressed as the product of two quadratic factors:

$$\begin{aligned} (I_{1y}I_{2y})s^4 + (I_{2y}C_2 + I_{1y}C_4)s^3 + (I_{1y}K_4 + C_2C_4 + I_{2y}K_2)s^2 + (C_2K_4 + C_4K_2)s \\ + K_2K_4 = (I_{1y}s^2 + C_2s + K_2)(I_{2y}s^2 + C_4s + K_4) \end{aligned} \quad (49)$$

A coarse approximation to the pitch roots is obtained from the roots of the two quadratic factors of equation (49).

## STABILITY ANALYSIS FOR $\bar{u} \neq \bar{0}$

Assume now that the central-body and solar-array centers of mass do not coincide ( $\bar{u} \neq \bar{0}$ ). If  $\{f\}$  in equation (6) is a nonzero matrix, a steady-state error  $\{\theta_{ss}\}$  will result. The steady-state error is

$$\{\theta_{ss}\} = [K]^{-1}\{f\} \quad (50)$$

The matrix  $\{f\}$ , as given by equation (10), will be nonzero only if the vector  $\bar{u}$  has more than one nonzero component in the body 1 coordinate system. To evaluate  $\{\theta_{ss}\}$  from equation (50), we must first find  $[K]^{-1}$ . Assuming that

$$K_i \gg \omega_0^2 \left[ I_{2z} + \frac{4M_1M_2}{M_T} |\bar{u}|^2 \right] \quad i = 1, 2, 3, 4$$

the matrix  $[K]$  can be approximated as

$$[K] = \begin{bmatrix} K_1 & 0 & 0 & 0 \\ -K_4 G \sin \varphi & K_2 - K_4 & -K_4 G \cos \varphi & -K_4 \\ 0 & 0 & K_3 & 0 \\ K_4 G \sin \varphi & K_4 & K_4 G \cos \varphi & K_4 \end{bmatrix} \quad (51)$$

Inverting the  $[K]$  matrix of equation (51) and carrying out the matrix multiplication of equation (50) yields

$$\{\theta_{ss}\} = \frac{M_1 M_2 \omega_0^2}{M_T} \left\{ \begin{array}{c} \frac{4u_{1y}u_{1z}}{K_1} \\ \frac{-3u_{1x}u_{1z}}{K_2} \\ \frac{-u_{1x}u_{1y}}{K_3} \\ \frac{-4u_{1y}u_{1z}G \sin \varphi}{K_1} + \frac{3u_{1x}u_{1z}}{K_2} + \frac{u_{1x}u_{1y}G \cos \varphi}{K_3} \end{array} \right\} \quad (52)$$

The steady-state solar-array pointing error  $\sigma_{ss}$  is calculated from equation (2) to be

$$\sigma_{ss} = 0 \quad (53)$$

To find the order of magnitude of  $\{\theta_{ss}\}$ , we assume the following values, which are representative of a high-power communication satellite configured as in figure 1:

$$\left. \begin{array}{ll} M_1 = 450 \text{ kg} & u_i = 1 \text{ m} \\ M_2 = 150 \text{ kg} & G = 0 \\ K_i = 1 \text{ Nm} & \varphi = 0 \end{array} \right\} \quad (54)$$

The values assumed for  $u_i$  represent a worst-case situation. For these values  $\{\theta_{ss}\}$  in degrees is

$$\{\theta_{ss}\} = \begin{bmatrix} 1.4 \times 10^{-4} \\ -1.0 \times 10^{-4} \\ -0.3 \times 10^{-4} \\ 1.0 \times 10^{-4} \end{bmatrix} \quad (55)$$

Thus, steady-state pointing errors due to nonzero  $\bar{u}$  are negligible.

To analyze stability for nonzero  $\bar{u}$ , we wish to make assumptions which allow the characteristic equation to be separated into three equations for the roll, yaw, and pitch axes. Paralleling the stability analysis for zero  $\bar{u}$ , the first step is to factor the characteristic equation into two quartic factors, where one factor contains roll and yaw parameters and the other factor contains pitch parameters. For the zero- $\bar{u}$  analysis, the two quartic factors are given by equations (20) and (21). In order to obtain this factorization it is necessary for  $z_{12}$ ,  $z_{14}$ ,  $z_{32}$ , and  $z_{34}$  to be zero. For zero  $\bar{u}$ , this is the case as seen by observing the zero elements of  $[M]$ ,  $[C]$ , and  $[K]$  in equations (7), (8), and (9). For nonzero  $\bar{u}$ ,  $z_{12}$  and  $z_{32}$  become nonzero. We make the assumption that the components of  $\bar{u}$  are small enough to justify setting  $z_{12}$  and  $z_{32}$  to zero. So the characteristic equation is still factorable into a roll-yaw characteristic equation,

$$z_{11}z_{33} - z_{13}z_{31} = 0 \quad (56)$$

and a pitch characteristic equation.

$$z_{22}z_{44} - z_{42}z_{24} = 0 \quad (57)$$

### Roll-Yaw Stability

Continuing to parallel the zero- $\bar{u}$  analysis, we wish to express the roll-yaw characteristic equation (56) as the product of two quadratic factors, where one factor contains roll parameters and the other factor contains yaw parameters. For the zero- $\bar{u}$  analysis, the factorization is given by equation (30). The required assumptions for making this factorization for nonzero  $\bar{u}$  are

$$\left| I_{2z} - I_{2x} \right| + \left| \frac{M_1 M_2}{M_T} u_{1x} u_{1z} \right| \ll I_{1x} + I_{2x} \quad (58)$$

$$|I_{2z} - I_{2x}| + \left| \frac{M_1 M_2}{M_T} u_{1x} u_{1z} \right| \ll I_{1z} + I_{2z} \quad (59)$$

$$C_i \gg \omega_o \left( 2I_{2x} + 2 \frac{M_1 M_2}{M_T} u_{1y}^2 \right) \quad i = 1, 3 \quad (60)$$

$$K_i \gg \omega_o^2 \left( I_{2x} + 4 \frac{M_1 M_2}{M_T} u_{1x} u_{1z} \right) \quad i = 1, 3 \quad (61)$$

These assumptions are equivalent to assuming that  $z_{13}$  and  $z_{31}$  are zero, and that

$$z_{11} = \left[ I_{1x} + I_{2x} + \frac{M_1 M_2}{M_T} (u_{1y}^2 + u_{1z}^2) \right] s^2 + C_1 s + K_1 = 0 \quad (62)$$

$$z_{33} = \left[ I_{1z} + I_{2z} + \frac{M_1 M_2}{M_T} (u_{1x}^2 + u_{1y}^2) \right] s^2 + C_3 s + K_3 = 0 \quad (63)$$

Equations (62) and (63) are the separated roll and yaw characteristic equations, respectively.

Define effective inertias  $I_x$  and  $I_z$  by the equations

$$I_x = I_{1x} + I_{2x} + \frac{M_1 M_2}{M_T} (u_{1y}^2 + u_{1z}^2) \quad (64)$$

$$I_z = I_{1z} + I_{2z} + \frac{M_1 M_2}{M_T} (u_{1x}^2 + u_{1y}^2) \quad (65)$$

Assuming that

$$C_1^2 - 4K_1 I_x < 0$$

$$C_3^2 - 4K_3 I_z < 0$$

then the roll roots are

$$s_{r1}, s_{r2} = \frac{-C_1}{2I_x} \pm j \frac{\sqrt{4K_1 I_x - C_1^2}}{2I_x} \quad (66)$$

and the yaw roots are

$$s_{y1}, s_{y2} = \frac{-C_3}{2I_z} \pm j \frac{\sqrt{4K_3 I_z - C_3^2}}{2I_z} \quad (67)$$

The real parts of the roll and yaw roots are negative for positive  $C_1$  and  $C_3$ . We conclude that stability is achieved in the roll and yaw axes for nonzero  $\bar{u}$ . Notice, however, that the degree of stability is less than for a zero  $\bar{u}$ . By comparing equations (66) and (67) with equations (35) and (36), it is seen that the real parts of the roll and yaw roots are smaller in magnitude for a nonzero  $\bar{u}$  than for a zero  $\bar{u}$ . For realistic values of  $\bar{u}$ , however, the difference in the magnitudes of the roots for the two cases is not substantial. The locus of the roll-yaw roots for nonzero  $\bar{u}$  is essentially the same as the locus of roll-yaw roots for zero  $\bar{u}$  given in figures 7 and 8.

### Pitch Stability

If we assume that

$$K_i \gg \frac{3M_1 M_2 \omega_0^2}{M_T} (u_{1z}^2 - u_{1x}^2) \quad (68)$$

then the pitch characteristic equation for a nonzero  $\bar{u}$  differs from the pitch characteristic equation for a zero  $\bar{u}$  in that the term  $I_{1y}$  is replaced by an effective pitch inertia  $I'_{1y}$  defined by

$$I'_{1y} = I_{1y} + \frac{M_1 M_2}{M_T} (u_{1x}^2 + u_{1y}^2) \quad (69)$$

From the results of the roll-yaw stability analysis, one might suspect that the effective pitch inertia  $I'_{1y}$  will decrease the magnitude of the real parts of the pitch characteristic roots and thus decrease the degree of stability. Defining  $\Delta I_{1y}$  by



$$\Delta I_{1y} = I'_{1y} - I_{1y} \quad (70)$$

we wish to show that the sum of the real parts of the pitch roots decreases in magnitude as  $\Delta I_{1y}$  increases. If the pitch characteristic equation is normalized so that the coefficient of  $s^4$  is 1, the coefficient of  $s^3$  becomes

$$\frac{I_{2y}(C_2 + C_4) + I_{1y}C_4 + \Delta I_{1y}C_4}{(I_{1y} + \Delta I_{1y})I_{2y}} \quad (71)$$

Calling this coefficient  $a$ , we obtain

$$\frac{da}{d(\Delta I_{1y})} = \frac{-I_{2y}^2(C_2 + C_4)}{(I_{1y} + \Delta I_{1y})^2 I_{2y}^2} \quad (72)$$

In terms of the pitch characteristic roots  $s_{pi}$ ,  $i = 1$  to  $4$ , the coefficient  $a$  is

$$a = -(s_{p1} + s_{p2} + s_{p3} + s_{p4})$$

Since the imaginary parts of the roots sum to zero,

$$a = -\text{Re}(s_{p1} + s_{p2} + s_{p3} + s_{p4}) \quad (73)$$

If it is assumed that  $C_2 + C_4$  is positive,  $da/d(\Delta I_{1y})$  is negative by equation (72). We conclude from equations (72) and (73) that the sum of the real parts of the pitch roots decreases in magnitude as  $\Delta I_{1y}$  increases. We cannot conclude, however, that pitch stability decreases as  $\Delta I_{1y}$  increases. It is possible that the dominant root (the root whose real part has the smallest magnitude) is increasing in magnitude at the same time that the sum of the real parts of the roots is decreasing in magnitude. Since the degree of stability is determined by the dominant root, in such a case the degree of stability increases as  $\Delta I_{1y}$  increases. Such a case is shown in the locus of pitch roots shown in figure 11. The locus parameter is  $\Delta I_{1y}/I_{1y}$  and the control gains and inertias are assumed to be

$$4I_{2y} = I_{1y} = 400 \text{ kg m}^2$$

$$4K_4 = K_2 = 40 \text{ Nm}$$

$$4C_4 = C_2 = 40 \text{ Nm sec}$$

It is seen from figure 11 that the real part of the dominant root increases in magnitude as  $\Delta I_{1y}/I_{1y}$  increases from 0 to 0.7. For realistic configurations of high-power communication satellites,  $\Delta I_{1y}/I_{1y}$  is not greater than 0.1. The change in magnitude of the real part of the dominant root as  $\Delta I_{1y}/I_{1y}$  changes from 0 to 0.1 is not significant. Thus, a nonzero  $\bar{u}$  does not significantly change stability characteristics, and the locus of pitch roots for nonzero  $\bar{u}$  is essentially the same as the locus of pitch roots for zero  $\bar{u}$  given in figures 9 and 10.

## CONCLUDING REMARKS

This report presents a small-angle stability analysis for one particular configuration of a high-power communication satellite having a linear control system. The results of the report yield preliminary indications of the relation of stability to satellite inertias and control gains.

Assumptions are made which allow the equations of motion to be linearized and which make the resulting stability analysis tractable. The eighth-order characteristic equation is separated into two second-order equations for the roll and yaw axes of the spacecraft and a fourth-order equation for the pitch axis. For positive control gains, stability is achieved for the roll and yaw axes. Not all positive values of control gains will yield stability for the pitch axis. Pitch-axis stability is achieved only when the control gains are sufficiently large. The degree of stability is not significantly changed when the central body and solar array have noncoincident centers of mass.

The exact roots of the eighth-order characteristic equation were calculated on a digital computer by using a subprogram which calculates the roots of a polynomial. The exact roots were compared to the roll, yaw, and pitch roots obtained from the approximate analysis. For the range of parameters which represent realistic high-power communication satellites, the approximate roots are within 2 percent of the exact roots.

Lewis Research Center,

National Aeronautics and Space Administration,

Cleveland, Ohio, April 25, 1972,

164-21.

## APPENDIX A

### DERIVATION OF GENERALIZED DYNAMIC EQUATIONS

#### Introduction

The generalized nonlinear equations of motion are derived in this appendix. The nomenclature and equation derivation is taken from reference 1. The equations as derived in reference 1 apply to a spacecraft configuration consisting of a central rigid body and as many as five auxiliary rigid bodies. Each auxiliary body is assumed to be connected to the central body at a single point, to have as many as three rotational degrees of freedom relative to the central body, and to have no interconnection with other auxiliary bodies. The equations presented in this appendix are a specialization of the equations of reference 1 for the case of one auxiliary body (solar array) having one rotational degree of freedom with respect to the central body and having the attach point between bodies at the center of mass of the auxiliary body.

#### Matrix Notation

It will be convenient to derive equations of motion in matrix notation. Brackets denote a square matrix and braces denote a column matrix. For example  $[A]$  is a square matrix and  $\{u_1\}$  is a column matrix. The notation  $[A]_T$  denotes the transpose of  $[A]$ . If  $\bar{\sigma}_i$  is a vector associated with body  $i$ , then  $\{\sigma_{ij}\}$  denotes the column matrix of components of  $\bar{\sigma}_i$  expressed in the body  $j$  coordinate system. Similarly,  $\{\sigma_{ir}\}$  denotes the column matrix of components of  $\sigma_i$  expressed in the orbital reference coordinate system. Those vectors, such as  $\bar{u}$ , which are not associated with either body 1 or body 2 do not have a subscript. For such a vector,  $\{u_j\}$  denotes the column matrix of components of  $\bar{u}$  expressed in the body  $j$  coordinate system. The components of a column matrix are denoted by additional subscripts  $x$ ,  $y$ , and  $z$ . Thus, the components of the column matrix  $\{\sigma_{ij}\}$  are  $\sigma_{ijx}$ ,  $\sigma_{ijy}$ , and  $\sigma_{ijz}$ .

Consider the vector cross product  $\bar{A} \times \bar{B}$ . We wish to express the cross product in matrix notation. Let  $\{A_j\}$  and  $\{B_j\}$  be the components of  $\bar{A}$  and  $\bar{B}$  in the body  $j$  coordinate system. Define  $[J_{Aj}]$  to be

$$[J_{Aj}] = \begin{bmatrix} 0 & -A_{jz} & A_{jy} \\ A_{jz} & 0 & -A_{jx} \\ -A_{jy} & A_{jx} & 0 \end{bmatrix} \quad (A1)$$

Then the components of  $(\bar{A} \times \bar{B})$  in the body  $j$  coordinate system are

$$\{(\bar{A} \times \bar{B})_j\} = [J_{Aj}]\{B_j\} \quad (A2)$$

Inertial derivatives of an arbitrary vector  $\bar{\alpha}$  will be denoted by  $d\alpha/dt$ ,  $d^2\alpha/dt^2$ ; and derivatives relative to a rotating coordinate system will be denoted by  $\dot{\bar{\alpha}}$ ,  $\ddot{\bar{\alpha}}$ .

## Coordinate Systems

Inertially fixed coordinate system. - The inertially fixed coordinate system will be denoted symbolically by

$$\{\bar{Q}_I\} = \left\{ \begin{array}{c} \bar{x}_I \\ \bar{y}_I \\ \bar{z}_I \end{array} \right\} \quad (A3)$$

where  $\bar{x}_I$ ,  $\bar{y}_I$ , and  $\bar{z}_I$  are unit vectors along the coordinate axes. The origin of the  $\{\bar{Q}_I\}$  coordinate system is at the center of the earth. The vectors  $\bar{x}_I$  and  $\bar{z}_I$  lie in the equatorial plane, and  $\bar{z}_I$  points to the sun at the time of the autumnal equinox. The  $\bar{y}_I$  axis points northward.

Orbital reference coordinate system. - The orbital reference coordinate system will be denoted symbolically by

$$\{\bar{Q}_r\} = \left\{ \begin{array}{c} \bar{x}_r \\ \bar{y}_r \\ \bar{z}_r \end{array} \right\} \quad (A4)$$

where  $\bar{x}_r$ ,  $\bar{y}_r$ , and  $\bar{z}_r$  are unit vectors along the coordinate axes. The origin of the  $\{\bar{Q}_r\}$  coordinate system is at the center of mass of the central body. The  $\bar{z}_r$  axis is directed toward the center of the earth, the  $\bar{x}_r$  axis is directed along the orbit velocity vector, and the  $\bar{y}_r$  axis points southward. It is assumed that the satellite orbit is exactly the circular synchronous equatorial orbit.

Body 1 coordinate system. - The body 1 (central body) coordinate system will be denoted symbolically by

$$\{\bar{Q}_1\} = \begin{Bmatrix} \bar{x}_1 \\ \bar{y}_1 \\ \bar{z}_1 \end{Bmatrix} \quad (A5)$$

where  $\bar{x}_1$ ,  $\bar{y}_1$ , and  $\bar{z}_1$  are unit vectors along the coordinate axes. The origin of the  $\{\bar{Q}_1\}$  coordinate system is at the center of mass of body 1. The axes of the coordinate system are oriented such that they coincide with the corresponding axes of  $\{\bar{Q}_r\}$  when body 1 is in its nominal position.

Body 2 coordinate system. - The body 2 (solar array) coordinate system will be denoted symbolically by

$$\{\bar{Q}_2\} = \begin{Bmatrix} \bar{x}_2 \\ \bar{y}_2 \\ \bar{z}_2 \end{Bmatrix} \quad (A6)$$

where  $\bar{x}_2$ ,  $\bar{y}_2$ , and  $\bar{z}_2$  are unit vectors along the coordinate axes. The origin of the  $\{\bar{Q}_2\}$  coordinate system is at the attach point connecting body 1 and body 2. The unit vector  $\bar{y}_2$  is parallel to  $\bar{y}_1$  and in the same direction as  $\bar{y}_1$ . Since body 2 has only one rotational degree of freedom with respect to body 1, rotation can take place only about the  $\bar{y}_2$ -axis. The unit vector  $\bar{z}_2$  is perpendicular to the solar array panels. The unit vector  $\bar{x}_2$  is in right-handed orientation to  $\bar{y}_2$  and  $\bar{z}_2$ .

## Relative Motion Between Bodies

Rotational positions. - Transformation matrices from one coordinate system to another are  $[E]$ ,  $[A]$ , and  $[D]$  as defined by the following equations.

$$\{\bar{Q}_r\} = [E]\{\bar{Q}_1\} \quad (A7)$$

$$\{\bar{Q}_1\} = [A]\{\bar{Q}_r\} \quad (A8)$$

$$\{\bar{Q}_2\} = [D]\{\bar{Q}_1\} \quad (A9)$$

Rotation velocities. - Rotational velocities of coordinate systems are now defined. The symbol  $\bar{\omega}$  is used for inertial rotational velocities, and the symbol  $\bar{\Omega}$  is used for relative rotational velocities. The following definitions will be used:

$\overline{\omega}_r, \overline{\Omega}_r$	inertial rotational velocity of orbital reference coordinate system
$\overline{\omega}_1$	inertial rotational velocity of body 1 coordinate system
$\overline{\Omega}_1$	rotational velocity of body 1 coordinate system with respect to orbital reference coordinate system
$\overline{\omega}_2$	inertial rotational velocity of body 2 coordinate system
$\overline{\Omega}_2$	rotational velocity of body 2 coordinate system with respect to body 1 coordinate system

The equations of motion will be developed to solve for the time rate of change of the relative rotational velocities  $\dot{\overline{\Omega}}_1$  and  $\dot{\overline{\Omega}}_2$ .

Rotational accelerations. - Rotational accelerations of coordinate systems are now defined. A dot over a column matrix variable indicates time derivative of the column matrix components. For example, the three components of  $\{\dot{\overline{\Omega}}_{rr}\}$  are

$$\frac{d\overline{\Omega}_{rrx}}{dt} \quad \frac{d\overline{\Omega}_{rry}}{dt} \quad \frac{d\overline{\Omega}_{rrz}}{dt}$$

The equations of motion will contain the following rotational accelerations:

$\{\dot{\overline{\omega}}_{rr}\} \equiv \{\dot{\overline{\Omega}}_{rr}\}$	inertial angular acceleration components of the orbital reference coordinate system
$\{\dot{\overline{\omega}}_{11}\}$	inertial angular acceleration components of the body 1 coordinate system
$\{\dot{\overline{\Omega}}_{11}\}$	angular acceleration components of the body 1 coordinate system with respect to the orbital reference coordinate system
$\{\dot{\overline{\omega}}_{22}\}$	inertial angular acceleration components of the body 2 coordinate system
$\{\dot{\overline{\Omega}}_{22}\}$	angular acceleration components of the body 2 coordinate system with respect to the body 1 coordinate system

We derive  $\{\dot{\overline{\omega}}_{11}\}$  as follows:

$$\overline{\omega}_1 = \overline{\omega}_r + \overline{\Omega}_1 \quad (A10)$$

$$\{\overline{\omega}_{11}\} = \{\overline{\omega}_{r1}\} + \{\overline{\Omega}_{11}\} = [A]\{\overline{\Omega}_{rr}\} + \{\overline{\Omega}_{11}\} \quad (A11)$$

$$\{\dot{\overline{\omega}}_{11}\} = [A]\{\dot{\overline{\Omega}}_{rr}\} + [\dot{A}]\{\overline{\Omega}_{rr}\} + \{\dot{\overline{\Omega}}_{11}\} \quad (A12)$$

Letting  $\omega_o$  be the angular velocity of the synchronous orbit, we have



$$\{\Omega_{rr}\} = \begin{Bmatrix} 0 \\ -\omega_0 \\ 0 \end{Bmatrix} \quad \{\dot{\Omega}_{rr}\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (\text{A13})$$

Equation (A12) becomes

$$\{\dot{\omega}_{11}\} = [\dot{A}]\{\Omega_{rr}\} + \{\dot{\Omega}_{11}\} \quad (\text{A14})$$

We derive  $\{\dot{\omega}_{22}\}$  as follows:

$$\bar{\omega}_2 = \bar{\omega}_1 + \bar{\Omega}_2 \quad (\text{A15})$$

$$\{\omega_{22}\} = \{\omega_{12}\} + \{\Omega_{22}\} = [D]\{\omega_{11}\} + \{\Omega_{22}\} \quad (\text{A16})$$

Using equation (A11), we obtain

$$\{\omega_{22}\} = [D][A]\{\Omega_{rr}\} + [D]\{\Omega_{11}\} + \{\Omega_{22}\} \quad (\text{A17})$$

Differentiating equation (A17), we obtain

$$\{\dot{\omega}_{22}\} = [\dot{D}][A]\{\Omega_{rr}\} + [D][\dot{A}]\{\Omega_{rr}\} + [\dot{D}]\{\Omega_{11}\} + [D]\{\dot{\Omega}_{11}\} + \{\dot{\Omega}_{22}\} \quad (\text{A18})$$

In developing equations of motion, we wish to isolate terms containing the relative rotational accelerations  $\{\dot{\Omega}_{11}\}$  and  $\{\dot{\Omega}_{22}\}$ . We let  $\{u\dot{\omega}_{11}\}$  be those terms of  $\{\dot{\omega}_{11}\}$  which contain  $\{\dot{\Omega}_{11}\}$  and  $\{\dot{\Omega}_{22}\}$ , and we let  $\{n\dot{\omega}_{11}\}$  be those terms of  $\{\dot{\omega}_{11}\}$  which do not contain  $\{\dot{\Omega}_{11}\}$  and  $\{\dot{\Omega}_{22}\}$ . We make similar definitions for  $\{u\dot{\omega}_{22}\}$  and  $\{n\dot{\omega}_{22}\}$ . Using equations (A14) and (A18), we have

$$\{u\dot{\omega}_{11}\} = \{\dot{\Omega}_{11}\} \quad (\text{A19})$$

$$\{n\dot{\omega}_{11}\} = [\dot{A}]\{\Omega_{rr}\} \quad (\text{A20})$$

$$\{u\dot{\omega}_{22}\} = [D]\{\dot{\Omega}_{11}\} + \{\dot{\Omega}_{22}\} \quad (\text{A21})$$

$$\{n\dot{\omega}_{22}\} = [\dot{D}][A]\{\Omega_{rr}\} + [D][\dot{A}]\{\Omega_{rr}\} + [\dot{D}]\{\Omega_{11}\} \quad (\text{A22})$$

Translational accelerations. - We now seek an expression for the translational acceleration of the body 2 center of mass expressed in body 1 coordinates. Let  $\bar{u}$  be the vector from the body 1 center of mass to the body 2 center of mass. Let  $\bar{S}_i$  be the vector from the center of the earth to the body i center of mass. Let  $\bar{S}_m$  be the vector from the center of the earth to the total spacecraft center of mass (fig. 12). We have the relation

$$\bar{S}_2 = \bar{S}_m + \frac{M_1}{M_1 + M_2} \bar{u} \quad (A23)$$

where  $M_1$  and  $M_2$  are the masses of body 1 and body 2, respectively. We define  $\bar{G}_m$ ,  $\bar{H}_u$ ,  $\bar{G}_2$  by the equations

$$\bar{G}_m = \frac{d^2 \bar{S}_m}{dt^2} \quad (A24a)$$

$$\bar{H}_u = \frac{d^2 \bar{u}}{dt^2} \quad (A24b)$$

$$\bar{G}_2 = \frac{d^2}{dt^2} \left( \bar{S}_m + \frac{M_1}{M_T} \bar{u} \right) = \frac{d^2 \bar{S}_2}{dt^2} \quad (A25)$$

where  $M_T$  is the sum of  $M_1$  and  $M_2$ . Thus,  $\bar{G}_2$  is the translational acceleration of the body 2 center of mass. Equation (A25) can be written as

$$\bar{G}_2 = \bar{G}_m + \frac{M_1}{M_T} \bar{H}_u \quad (A26)$$

The vector  $\bar{H}_u$  can be written as

$$\bar{H}_u = \ddot{\bar{u}} + 2(\bar{\omega}_1 \times \dot{\bar{u}}) + (\dot{\bar{\omega}}_1 \times \bar{u}) + \bar{\omega}_1 \times (\bar{\omega}_1 \times \bar{u}) \quad (A27)$$

where the derivatives on the right-hand side of the equation are taken with respect to the body 1 coordinate system.

In matrix notation, equation (A27) in the body 1 coordinate system becomes

$$\{ \ddot{H}_{u1} \} = \{ \ddot{u}_1 \} + 2[J_{\omega 11}] \{ \dot{u}_1 \} + [J_{\dot{\omega} 11}] \{ u_1 \} + [J_{\omega 11}][J_{\omega 11}] \{ u_1 \} \quad (A28)$$

Since the vector  $\bar{u}$  has fixed components in the body 1 coordinate system,

$$\{\ddot{u}_1\} = \{\dot{u}_1\} = \{0\} \quad (A29)$$

Equation (A28) simplifies to

$$\{H_{u1}\} = [J_{\dot{\omega}11}]\{u_1\} + [J_{\omega11}][J_{\omega11}]\{u_1\} \quad (A30)$$

$$\{H_{u1}\} = -[J_{u1}]\{\dot{\omega}_{11}\} + [J_{\omega11}][J_{\omega11}]\{u_1\} \quad (A31)$$

Separating equation (A31) into two components,  $\{^n H_{u1}\}$  and  $\{^u H_{u1}\}$ ,

$$\{^n H_{u1}\} = -[J_{u1}]\{^n \dot{\omega}_{11}\} + [J_{\omega11}][J_{\omega11}]\{u_1\} \quad (A32)$$

$$\{^u H_{u1}\} = -[J_{u1}]\{^u \dot{\omega}_{11}\} \quad (A33)$$

## Forces and Torques

In discussing forces and torques, we will use the term "internal torque" to refer to reaction torques between the two bodies. All other torques will be termed "external." We make the following definitions:

$\bar{F}_{gi}$	gravity force on body i, $i = 1, 2$
$\bar{F}_{si}$	external forces, excluding gravity force, on body i
$\bar{F}_r$	internal reaction force on body 1 due to body 2
$\bar{T}_{gi}$	gravity gradient torque (about body i center of mass) on body i, $i = 1, 2$
$\bar{T}_{si}$	external torques (about body i center of mass) excluding gravity gradient torque, on body i, $i = 1, 2$
$\bar{T}_r$	internal reaction torque (about body 2 center of mass) on body 1 due to body 2

The total force on body 1 in vector and matrix notation is

$$\bar{F}_{e1} = \bar{F}_{g1} + \bar{F}_{s1} + \bar{F}_r \quad (A34)$$

$$\{F_{e11}\} = \{F_{g11}\} + \{F_{s11}\} + \{F_{r1}\} \quad (A35)$$

The total torque (about the body 1 center of mass) on body 1 in vector and matrix notation is

$$\bar{T}_{e1} = \bar{T}_{g1} + \bar{T}_{s1} + (\bar{u} \times \bar{F}_r) + \bar{T}_r \quad (A36)$$

$$\{T_{e11}\} = \{T_{g11}\} + \{T_{s11}\} + [J_{u1}][D]_T \{F_{r2}\} + [D]_T \{T_{r2}\} \quad (A37)$$

The total force on body 2 in vector and matrix notation is

$$\bar{F}_{e2} = \bar{F}_{g2} + \bar{F}_{s2} - \bar{F}_r \quad (A38)$$

$$\{F_{e22}\} = \{F_{g22}\} + \{F_{s22}\} - \{F_{r2}\} \quad (A39)$$

The total torque (about the body 2 center of mass) on body 2 in vector and matrix notation is

$$\bar{T}_{e2} = \bar{T}_{g2} + \bar{T}_{s2} - \bar{T}_r \quad (A40)$$

$$\{T_{e22}\} = \{T_{g22}\} + \{T_{s22}\} - \{T_{r2}\} \quad (A41)$$

The total force on body 2 is equal to the product of its mass and the acceleration of its center of mass

$$\bar{F}_{e2} = M_2 \bar{G}_2 \quad (A42)$$

Using equations (A38) and (A42), we have

$$\bar{F}_r = \bar{F}_{g2} + \bar{F}_{s2} - M_2 \bar{G}_2 \quad (A43)$$

### Body 1 Equation of Motion

The body 1 equation of motion in vector and tensor notation is

$$\bar{T}_{e1} = \bar{I}_1 \cdot \dot{\bar{\omega}}_1 + \bar{\omega}_1 \times \bar{I}_1 \cdot \bar{\omega}_1 \quad (A44)$$

where  $\dot{\bar{\omega}}_1$  denotes the derivative of  $\bar{\omega}_1$  with respect to the body 1 coordinate system, and  $\bar{I}_1$  is the inertia tensor of body 1.

Expand  $\bar{T}_{e1}$  using equations (A36) and (A43):

$$\bar{T}_{e1} = \bar{T}_{g1} + \bar{T}_{s1} + \bar{u} \times (\bar{F}_{g2} + \bar{F}_{s2} - M_2 \bar{G}_2) + \bar{T}_r \quad (A45)$$

$$\bar{T}_{e1} = \bar{T}_{g1} + \bar{T}_{s1} + \bar{u} \times (\bar{F}_{g2} - M_2 \bar{G}_2) + \bar{u} \times \bar{F}_{s2} + \bar{T}_r \quad (A46)$$

The term  $\bar{F}_{g2} - M_2 \bar{G}_2$  is the subtraction of two nearly equal terms. We wish to express this term in a more desirable form. Using equation (A23),  $\bar{F}_{g2}$  can be written as

$$\bar{F}_{g2} = -\frac{\gamma M_2 \bar{S}_2}{|\bar{S}_2|^3} = -\frac{\gamma M_2 \left( \bar{S}_m + \frac{M_1}{M_T} \bar{u} \right)}{\left| \bar{S}_m + \frac{M_1}{M_T} \bar{u} \right|^3} \quad (A47)$$

Since  $\bar{S}_m$  is much larger than  $(M_1/M_T)\bar{u}$ , we can use a perturbation technique to express  $\bar{F}_{g2}$  as

$$\bar{F}_{g2} = -\gamma M_2 \left[ \frac{\bar{S}_m}{S_m^3} - \frac{3 \left( \bar{S}_m \cdot \frac{M_1}{M_T} \bar{u} \right) \bar{S}_m}{S_m^5} + \frac{M_1 \bar{u}}{M_T S_m^3} \right] \quad (A48)$$

We calculate  $\bar{G}_m$  from

$$M_T \bar{G}_m = \sum_{i=1}^2 \bar{F}_{gi} + \sum_{i=1}^2 \bar{F}_{si} = -\frac{\gamma M_T \bar{S}_m}{S_m^3} + \sum_{i=1}^2 \bar{F}_{si} \quad (A49)$$

Equation (A49) reduces to

$$\bar{G}_m = -\frac{\gamma \bar{S}_m}{S_m^3} + \frac{1}{M_T} \sum_{i=1}^2 F_{si} \quad (A50)$$

Then  $M_2 \bar{G}_2$  is obtained by using equation (A26)

$$M_2 \bar{G}_2 = M_2 \bar{G}_m + \frac{M_1 M_2}{M_T} \bar{H}_u \quad (A51)$$

Now using equation (A50),

$$M_2 \bar{G}_2 = -\frac{\gamma M_2 \bar{S}_m}{S_m^3} + \frac{M_2}{M_T} \sum_{i=1}^2 \bar{F}_{si} + \frac{M_1 M_2}{M_T} \bar{H}_u \quad (A52)$$

Subtracting equation (A52) from equation (A48), we obtain the term  $(\bar{F}_{g2} - M_2 \bar{G}_2)$  in a more desirable form.

$$\bar{F}_{g2} - M_2 \bar{G}_2 = -\gamma M_2 \left[ -\frac{3M_1(\bar{S}_m \cdot \bar{u})\bar{S}_m}{M_T S_m^5} + \frac{M_1 \bar{u}}{M_T S_m^3} \right] - \frac{M_2}{M_T} \sum_{i=1}^2 \bar{F}_{si} - \frac{M_1 M_2}{M_T} \bar{H}_u \quad (A53)$$

Combining equations (A46) and (A53) yields

$$\bar{T}_{e1} = \bar{T}_{g1} + \bar{T}_{s1} + \bar{u} \times \left[ \frac{3\gamma M_1 M_2 (\bar{S}_m \cdot \bar{u}) \bar{S}_m}{M_T S_m^5} - \frac{M_2}{M_T} \sum_{i=1}^2 \bar{F}_{si} - \frac{M_1 M_2 \bar{H}_u}{M_T} \right] + \bar{u} \times \bar{F}_{s2} + \bar{T}_r \quad (A54)$$

Using the vector identity

$$(\bar{u} \cdot \bar{S}_m)(\bar{u} \times \bar{S}_m) = \bar{u} \times [\bar{S}_m \times (\bar{S}_m \times \bar{u})] \quad (A55)$$

equation (A54) becomes

$$\begin{aligned} \bar{T}_{e1} = \bar{T}_{g1} + \bar{T}_{s1} + \bar{u} \times \left( \bar{F}_{s2} - \frac{M_2}{M_T} \sum_{i=1}^2 \bar{F}_{si} \right) + \bar{T}_r \\ + \frac{3\gamma M_1 M_2 \left\{ \bar{u} \times [\bar{S}_m \times (\bar{S}_m \times \bar{u})] \right\}}{S_m^3 M_T} - \frac{M_1 M_2 (\bar{u} \times \bar{H}_u)}{M_T} \end{aligned} \quad (A56)$$

We now wish to write the vector equation (A56) in matrix notation. The components of the vector  $\bar{S}_m$  in the orbital reference coordinate system are given by



$$\{S_{mr}\} = \begin{Bmatrix} 0 \\ 0 \\ -S_m \end{Bmatrix} \quad (A57)$$

where  $S_m$  is the magnitude of  $\bar{S}_m$ . Then

$$\{S_{m1}\} = -S_m [A] \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = -S_m \{A_3\} \quad (A58)$$

where  $\{A_3\}$  denotes the third column of the matrix  $[A]$ . We can write equation (A56) in the body 1 coordinate system as

$$\begin{aligned} \{T_{e11}\} = \{T_{g11}\} + \{T_{s11}\} + [J_{u1}] & \left( -\frac{M_2}{M_T} \{F_{s11}\} + \frac{M_1}{M_T} \{F_{s21}\} \right) \\ & + \frac{3\gamma M_1 M_2}{S_m^3 M_T} [J_{u1}] [J_{A3}] [J_{A3}] \{u_1\} - \frac{M_1 M_2}{M_T} [J_{u1}] \left( \{ {}_n H_{u1} \} + \{ {}_u H_{u1} \} \right) \end{aligned} \quad (A59)$$

Rewriting equation (A44) in matrix notation,

$$[I_1] \left( \{ {}_n \dot{\omega}_{11} \} + \{ {}_u \dot{\omega}_{11} \} \right) + [J_{\omega 11}] [I_1] \{ \omega_{11} \} = \{ T_{e11} \} \quad (A60)$$

where  $[I_1]$  is the inertia matrix of body 1 with respect to the body 1 coordinate system.

Combining equations (A59) and (A60)

$$[I_1] \{ {}_u \dot{\omega}_{11} \} + \frac{M_1 M_2}{M_T} [J_{u1}] \{ {}_u H_{u1} \} = \{ q_{11} \} + \{ T_{r1} \} \quad (A61)$$

where  $\{ q_{11} \}$  is defined to be

$$\begin{aligned}
\{q_{11}\} = & -[I_1]\{\dot{\omega}_{11}\} - [J_{\omega 11}][I_1]\{\omega_{11}\} + \{T_{g11}\} + \{T_{s11}\} \\
& + [J_{u1}]\left(-\frac{M_2}{M_T}\{F_{s11}\} + \frac{M_1}{M_T}\{F_{s21}\}\right) + \frac{3\gamma M_1 M_2}{S_m^3 M_T}[J_{u1}][J_{A3}][J_{A3}]\{u_1\} \\
& - \frac{M_1 M_2}{M_T}[J_{u1}]\{nH_{u1}\}
\end{aligned} \tag{A62}$$

By using equations (A19) and (A32), equation (A61) becomes

$$[I_1]\{\dot{\omega}_{11}\} - \frac{M_1 M_2}{M_T}[J_{u1}][J_{u1}]\{\dot{\omega}_{11}\} = \{q_{11}\} + \{T_{r1}\} \tag{A63}$$

In the analysis to follow, it will be convenient to express equation (A63) as

$$[L_{11}]\{\dot{\omega}_{11}\} + [L_{12}]\{\dot{\omega}_{22}\} = \{q_{11}\} + \{T_{r1}\} \tag{A64}$$

where  $[L_{11}]$  and  $[L_{12}]$  are given by

$$[L_{11}] = [I_1] - \frac{M_1 M_2}{M_T}[J_{u1}][J_{u1}] \tag{A65}$$

$$[L_{12}] = [0] \tag{A66}$$

## Body 2 Equation of Motion

The body 2 equation of motion in vector notation is

$$\bar{T}_{e2} = \bar{\bar{I}}_2 \cdot \dot{\bar{\omega}}_2 + \bar{\omega}_2 \times \bar{\bar{I}}_2 \cdot \bar{\omega}_2 \tag{A67}$$

where  $\dot{\bar{\omega}}_2$  denotes the derivative of  $\bar{\omega}_2$  with respect to the body 2 coordinate system and  $\bar{\bar{I}}_2$  is the inertia tensor of body 2. From equation (A40),  $\bar{T}_{e2}$  is given by

$$\bar{T}_{e2} = \bar{T}_{g2} + \bar{T}_{s2} - \bar{T}_r \tag{A68}$$

Combining equations (A67) and (A68) in a matrix equation in the body 2 coordinate system,

$$[I_2] \left( \{ \dot{\omega}_{22} \}_n + \{ \dot{\omega}_{22} \}_u \right) + [J_{\omega 22}][I_2] \{ \omega_{22} \} = \{ T_{g22} \} + \{ T_{s22} \} - \{ T_{r2} \} \quad (A69)$$

where  $[I_2]$  is the inertia matrix of body 2 with respect to the body 2 coordinate system.

By using equation (A21), equation (A69) becomes

$$[I_2] \left( [D] \{ \dot{\Omega}_{11} \} + \{ \dot{\Omega}_{22} \} \right) = \{ q_{22} \} - \{ T_{r2} \} \quad (A70)$$

where  $\{ q_{22} \}$  is given by

$$\{ q_{22} \} = -[I_2] \{ \dot{\omega}_{22} \}_n - [J_{\omega 22}][I_2] \{ \omega_{22} \} + \{ T_{g22} \} + \{ T_{s22} \} \quad (A71)$$

Equation (A70) is rewritten as

$$[L_{21}] \{ \dot{\Omega}_{11} \} + [L_{22}] \{ \dot{\Omega}_{22} \} = \{ q_{22} \} - \{ T_{r2} \} \quad (A72)$$

where  $[L_{21}]$  and  $[L_{22}]$  are given by

$$[L_{21}] = [I_2][D] \quad (A73)$$

$$[L_{22}] = [I_2] \quad (A74)$$

## Elimination of Locked-Gimbal Reaction Torques from Equations of Motion

We repeat the body 1 and body 2 equations of motion

$$[L_{11}] \{ \dot{\Omega}_{11} \} + [L_{12}] \{ \dot{\Omega}_{22} \} = \{ q_{11} \} + [D]_T \{ T_{r2} \} \quad (A75)$$

$$[L_{21}] \{ \dot{\Omega}_{11} \} + [L_{22}] \{ \dot{\Omega}_{22} \} = \{ q_{22} \} - \{ T_{r2} \} \quad (A76)$$

Body 2 has rotational freedom only about its y-axis. The x-axis and z-axis of body 2 are locked gimbals. Then  $\{ T_{r2} \}$  is composed of two parts

$$\{ T_{r2} \} = \{ T_{FG2} \} + \{ T_{LG2} \} \quad (A77)$$

where  $\{ T_{FG2} \}$  is the reaction torque about the "free-gimbal" y-axis, and  $\{ T_{LG2} \}$  is the reaction torque about the "locked-gimbal" x-axis and z-axis.  $\{ T_{FG2} \}$  has only a y component, and  $\{ T_{LG2} \}$  has only x and z components.

$$\{T_{FG2}\} = \begin{bmatrix} 0 \\ T_{FG2y} \\ 0 \end{bmatrix} \quad (A78)$$

$$\{T_{LG2}\} = \begin{bmatrix} T_{LG2x} \\ 0 \\ T_{LG2z} \end{bmatrix} \quad (A79)$$

Substituting equation (A77) into equations (A75) and (A76),

$$[L_{11}]\{\dot{\Omega}_{11}\} + [L_{12}]\{\dot{\Omega}_{22}\} = \{q_{11}\} + [D]_T\{T_{FG2}\} + [D]_T\{T_{LG2}\} \quad (A80)$$

$$[L_{21}]\{\dot{\Omega}_{11}\} + [L_{22}]\{\dot{\Omega}_{22}\} = \{q_{22}\} - \{T_{FG2}\} - \{T_{LG2}\} \quad (A81)$$

We now manipulate the body 1 and body 2 equations (A80) and (A81) in order to eliminate  $\{T_{LG2}\}$ . It will be convenient to introduce three new matrices,  $[U]$ ,  $[P]$ , and  $[U - P]$ , defined by

$$[U] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (A82)$$

$$[P] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (A83)$$

$$[U - P] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (A84)$$

We have the following relation:

$$[P]\{T_{FG2}\} = \{T_{FG2}\} \quad [U - P]\{T_{FG2}\} = \{0\} \quad (A85)$$

$$[P]\{T_{LG2}\} = \{0\} \quad [U - P]\{T_{LG2}\} = \{T_{LG2}\} \quad (A86)$$

To eliminate  $\{T_{LG2}\}$  from the body 2 equation of motion, we premultiply equation (A81) by  $[P]$ .

$$[P][L_{21}]\{\dot{\Omega}_{11}\} + [P][L_{22}]\{\dot{\Omega}_{22}\} = [P]\{q_{22}\} - \{T_{FG2}\} \quad (A87)$$

To eliminate  $\{T_{LG2}\}$  from the body 1 equation of motion, we must first premultiply the body 2 equation (A81) by  $[D]_T[U - P]$ .

$$[D]_T[U - P][L_{21}]\{\dot{\Omega}_{11}\} + [D]_T[U - P][L_{22}]\{\dot{\Omega}_{22}\} = [D]_T[U - P]\{q_{22}\} - [D]_T\{T_{LG2}\} \quad (A88)$$

By adding equation (A88) to the body 1 equation (A80), we obtain the new body 1 equation with  $\{T_{LG2}\}$  eliminated.

$$\begin{aligned} ([L_{11}] + [D]_T[U - P][L_{21}])\{\dot{\Omega}_{11}\} + ([L_{12}] + [D]_T[U - P][L_{22}])\{\dot{\Omega}_{22}\} &= \{q_{11}\} \\ &+ [D]_T\{T_{FG2}\} + [D]_T[U - P]\{q_{22}\} \end{aligned} \quad (A89)$$

Rewriting the new body 1 and body 2 equations (A89) and (A87),

$$[M_{11}]\{\dot{\Omega}_{11}\} + [M_{12}]\{\dot{\Omega}_{22}\} = \{b_{11}\} \quad (A90)$$

$$[M_{21}]\{\dot{\Omega}_{11}\} + [M_{22}]\{\dot{\Omega}_{22}\} = \{b_{22}\} \quad (A91)$$

where  $[M_{11}]$ ,  $[M_{12}]$ ,  $[M_{21}]$ ,  $[M_{22}]$ ,  $\{b_{11}\}$ , and  $\{b_{22}\}$  are given by

$$[M_{11}] = [L_{11}] + [D]_T[U - P][L_{21}] \quad (A92)$$

$$[M_{12}] = [L_{12}] + [D]_T[U - P][L_{22}] \quad (A93)$$

$$[M_{21}] = [P][L_{21}] \quad (A94)$$

$$[M_{22}] = [P][L_{22}] \quad (A95)$$

$$\{b_{11}\} = \{q_{11}\} + [D]_T \{T_{FG2}\} + [D]_T [U - P] \{q_{22}\} \quad (A96a)$$

$$\{b_{22}\} = [P] \{q_{22}\} - \{T_{FG2}\} \quad (A96b)$$

Since  $[M_{21}]$ ,  $[M_{22}]$ ,  $\{b_{22}\}$  all involve a premultiplication by  $[P]$ , the first and third rows of these three matrices are zeros. Also, the first and third components of  $\{\dot{\Omega}_{22}\}$  are zero because body 2 has rotation only about its y-axis. Combining equations (A90) and (A91) into a single matrix equation, we have

$$\begin{bmatrix} M_{11xx} & M_{11xy} & M_{11xz} & M_{12xx} & M_{12xy} & M_{12xz} \\ M_{11yx} & M_{11yy} & M_{11yz} & M_{12yx} & M_{12yy} & M_{12yz} \\ M_{11zx} & M_{11zy} & M_{11zz} & M_{12zx} & M_{12zy} & M_{12zz} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ M_{21yx} & M_{21yy} & M_{21yz} & M_{22yx} & M_{22yy} & M_{22yz} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\Omega}_{11x} \\ \dot{\Omega}_{11y} \\ \dot{\Omega}_{11z} \\ 0 \\ \dot{\Omega}_{22y} \\ 0 \end{Bmatrix} = \begin{Bmatrix} b_{11x} \\ b_{11y} \\ b_{11z} \\ 0 \\ b_{22y} \\ 0 \end{Bmatrix} \quad (A97)$$

The third and fifth rows and columns of the 6x6 coefficient matrix can be deleted to form a 4x4 system

$$\begin{bmatrix} M_{11xx} & M_{11xy} & M_{11xz} & M_{12xy} \\ M_{11yx} & M_{11yy} & M_{11yz} & M_{12yy} \\ M_{11zx} & M_{11zy} & M_{11zz} & M_{12zy} \\ M_{21yx} & M_{21yy} & M_{21yz} & M_{22yy} \end{bmatrix} \begin{Bmatrix} \dot{\Omega}_{11x} \\ \dot{\Omega}_{11y} \\ \dot{\Omega}_{11z} \\ \dot{\Omega}_{22y} \end{Bmatrix} = \begin{Bmatrix} b_{11x} \\ b_{11y} \\ b_{11z} \\ b_{22y} \end{Bmatrix} \quad (A98)$$

For shorthand notation, equation (A98) will be written as

$$[M] \{\dot{\Omega}\} = \{b\} \quad (A99)$$

## APPENDIX B

### LINEARIZATION OF DYNAMIC EQUATIONS

#### Assumptions

The dynamic equations derived in appendix A will now be linearized by assuming that the angular coordinates  $\theta_i$  are small. Thus,  $\sin \theta_i$  is approximated by  $\theta_i$ , and  $\cos \theta_i$  is approximated by 1. We make the following assumptions:

(1) The body 1 coordinate axes are nearly the same as the orbital reference coordinate axes. Thus, we can specify the orientation of the body 1 coordinate system with respect to the orbital reference coordinate system by three small angles  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$ . The orbital reference coordinate axes, if rotated through an angle  $\theta_1$  about the  $x_r$ -axis, through an angle  $\theta_2$  about the  $y_r$ -axis, and through an angle  $\theta_3$  about the  $z_r$ -axis, will coincide with the body 1 coordinate axes. The transformation matrix  $[A]$  becomes

$$[A] = \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix} \quad (B1)$$

(2) The body 1 and body 2 inertia matrices are diagonal matrices. Equivalently, the geometric axes coincide with the principal axes of bodies 1 and 2. The inertia matrices  $[I_1]$  and  $[I_2]$  become

$$[I_1] = \begin{bmatrix} I_{1x} & 0 & 0 \\ 0 & I_{1y} & 0 \\ 0 & 0 & I_{1z} \end{bmatrix} \quad (B2)$$

$$[I_2] = \begin{bmatrix} I_{2x} & 0 & 0 \\ 0 & I_{2y} & 0 \\ 0 & 0 & I_{2z} \end{bmatrix} \quad (B3)$$

(3) The body 1 and body 2 inertia matrices have constant elements, and the body 1 and body 2 masses are constant.

(4) The dominant torques on the spacecraft are the control torques. Environmental torques such as the gravity gradient, solar pressure, and magnetic torques are negligible. Any friction torque between the two bodies is negligible. It is assumed that  $\theta_i$  and  $\dot{\theta}_i$ ,  $i = 1, 2, 3$  are known from sensor data and that the components of the body 1 control torque vector  $\bar{T}_c$  consist of proportional and derivative feedback. The vector  $\bar{T}_c$  is given in the body 1 coordinate system by

$$\{T_{c1}\} = \begin{Bmatrix} -K_1\theta_1 - C_1\dot{\theta}_1 \\ -K_2\theta_2 - C_2\dot{\theta}_2 \\ -K_3\theta_3 - C_3\dot{\theta}_3 \end{Bmatrix} \quad (B4)$$

(5) A motor torque is used for rotating body 2 with respect to body 1. See figure 3 for the definition of the angle  $\varphi$ . The angular velocity of the body 2 coordinate system with respect to the body 1 coordinate system is  $\dot{\varphi}$ . The nominal value of  $\dot{\varphi}$  is one earth rate  $\omega_0$ . So the rate error is given by  $\dot{\varphi} - \omega_0$ . It is assumed that the motor torque responds to the position error  $\sigma$ , as defined in figure 4, and the rate error  $\dot{\varphi} - \omega_0$ . The motor torque vector  $\bar{T}_m$  is given in the body 2 coordinate system by

$$\{T_{m2}\} = \begin{Bmatrix} 0 \\ K_4\sigma + C_4(\dot{\varphi} - \omega_0) \\ 0 \end{Bmatrix} \quad (B5)$$

where it is assumed  $\bar{T}_m$  acts on body 1.

(6) Nonlinear terms in the equations of motion are negligible. Thus, terms such as  $\theta_i^2$ ,  $\theta_i\theta_j$ , and  $\theta_i\dot{\theta}_j$  are removed from the equations of motion.

### Calculation of $\sigma$

The fourth angular coordinate  $\theta_4$  is defined as

$$\theta_4 = \varphi - (\alpha + \omega_0 t) \quad (B6)$$

where the angles  $\alpha$  and  $\omega_0 t$  are shown in figure 5.

The body 2 pointing error  $\sigma$  is equal to  $\theta_4$  when  $\theta_1 = \theta_2 = \theta_3 = 0$ . When body 1 pointing errors are nonzero, they contribute to  $\sigma$ . Let  $\bar{\delta}$  be a unit vector directed from



the center of the earth to the sun. Let  $\bar{\delta}_p$  be a vector formed from the projection of  $\bar{\delta}$  into the  $\bar{z}_I$ - $\bar{x}_I$  plane. Let  $\theta_s$  be the angle between  $\bar{z}_I$  and  $\bar{\delta}$ , and let  $\theta_p$  be the angle between  $\bar{z}_I$  and  $\bar{\delta}_p$ . The geometry is shown in figure 13.

The angles  $\theta_s$  and  $\theta_p$  are related by

$$\tan \theta_s \cos i = \tan \theta_p \quad (\text{B7a})$$

where  $i$  is the inclination of the ecliptic plane to the equatorial plane ( $i = 23.449^\circ$ ). In the inertial coordinate system,  $\bar{\delta}$  is given by

$$\{\delta_I\} = \begin{Bmatrix} \sin \theta_s \cos i \\ -\sin \theta_s \sin i \\ \cos \theta_s \end{Bmatrix} \quad (\text{B7b})$$

A series of transformation matrices is required to express  $\bar{\delta}$  in the body 2 coordinate system.

$$\{\delta_2\} = [D][A][E]\{\delta_I\} \quad (\text{B8})$$

The  $[A]$  matrix is given by equation (B1), and  $[D]$  and  $[E]$  are given by

$$[D] = \begin{bmatrix} \cos \varphi & 0 & -\sin \varphi \\ 0 & 1 & 0 \\ \sin \varphi & 0 & \cos \varphi \end{bmatrix} \quad (\text{B9})$$

$$[E] = \begin{bmatrix} -\cos(\theta_p + \alpha_1 + \omega_o t) & 0 & \sin(\theta_p + \alpha_1 + \omega_o t) \\ 0 & -1 & 0 \\ \sin(\theta_p + \alpha_1 + \omega_o t) & 0 & \cos(\theta_p + \alpha_1 + \omega_o t) \end{bmatrix} \quad (\text{B10})$$

The angle  $\sigma$  is assumed small, and the sign convention for  $\sigma$  is that it is negative if the projection of  $\bar{\delta}$  into the  $\bar{z}_2$ - $\bar{x}_2$  plane lies between the  $\bar{z}_2$ -axis and  $\bar{x}_2$ -axis. Thus,

$$\sigma = \sin \sigma = \frac{-\delta_{2x}}{\sqrt{\delta_{2x}^2 + \delta_{2z}^2}} \quad (\text{B11})$$

Using equations (B6) to (B11),  $\sigma$  is calculated to be

$$\sigma = [G \sin(\varphi - \theta_4)]\theta_1 + \theta_2 + G[\cos(\varphi - \theta_4)]\theta_3 + \theta_4 \quad (\text{B12})$$

where  $G$  is given by

$$G = \frac{-\sin \theta_s \sin i}{\sqrt{1 - \sin^2 \theta_s \sin^2 i}} \quad (\text{B13})$$

Since  $\theta_4$  is a small angle and  $\varphi$  in general is not small, we can approximate equation (B12) by

$$\sigma = (G \sin \varphi)\theta_1 + \theta_2 + (G \cos \varphi)\theta_3 + \theta_4 \quad (\text{B14a})$$

The motor torque of equation (B5) can now be written as

$$\{T_{m2}\} = \begin{Bmatrix} 0 \\ (K_4 G \sin \varphi)\theta_1 + K_4 \theta_2 + (K_4 G \cos \varphi)\theta_3 + K_4 \theta_4 + C_4 \dot{\theta}_4 \\ 0 \end{Bmatrix} \quad (\text{B14b})$$

### Calculation of $[M]$

The matrix  $[M]$  of equation (A99) consists of elements of the matrices  $[M_{11}]$ ,  $[M_{12}]$ ,  $[M_{21}]$ , and  $[M_{22}]$ , which in turn consist of elements of the matrices  $[L_{11}]$ ,  $[L_{12}]$ ,  $[L_{21}]$ , and  $[L_{22}]$ . Evaluating  $[L_{11}]$ ,  $[L_{12}]$ ,  $[L_{21}]$ , and  $[L_{22}]$  from equations (A65), (A66), (A73), and (A74),

$$[L_{11}] = \begin{bmatrix} I_{1x} & 0 & 0 \\ 0 & I_{1y} & 0 \\ 0 & 0 & I_{1z} \end{bmatrix} + \frac{M_1 M_2}{M_T} \begin{bmatrix} u_{1y}^2 + u_{1z}^2 & -u_{1x}u_{1y} & -u_{1x}u_{1z} \\ -u_{1y}u_{1x} & u_{1z}^2 + u_{1y}^2 & -u_{1y}u_{1z} \\ -u_{1z}u_{1x} & -u_{1z}u_{1y} & u_{1x}^2 + u_{1y}^2 \end{bmatrix} \quad (\text{B15})$$

$$[L_{12}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{B16})$$

$$[L_{21}] = \begin{bmatrix} I_{2x} \cos \varphi & 0 & -I_{2x} \sin \varphi \\ 0 & I_{2y} & 0 \\ I_{2z} \sin \varphi & 0 & I_{2z} \cos \varphi \end{bmatrix} \quad (B17)$$

$$[L_{22}] = \begin{bmatrix} I_{2x} & 0 & 0 \\ 0 & I_{2y} & 0 \\ 0 & 0 & I_{2z} \end{bmatrix} \quad (B18)$$

Evaluating  $[M_{11}]$ ,  $[M_{12}]$ ,  $[M_{21}]$ , and  $[M_{22}]$  from equations (A92) to (A95),

$$[M_{11}] = \begin{bmatrix} I_{1x} + I_{2x} + (I_{2z} - I_{2x}) \sin^2 \varphi & 0 & (I_{2z} - I_{2x}) \sin \varphi \cos \varphi \\ 0 & I_{1y} & 0 \\ (I_{2z} - I_{2x}) \sin \varphi \cos \varphi & 0 & I_{1z} + I_{2z} - (I_{2z} - I_{2x}) \sin^2 \varphi \end{bmatrix} \\ + \frac{M_1 M_2}{M_T} \begin{bmatrix} u_{1y}^2 + u_{1z}^2 & -u_{1x} u_{1y} & -u_{1x} u_{1z} \\ -u_{1y} u_{1x} & u_{1z}^2 + u_{1y}^2 & -u_{1y} u_{1z} \\ -u_{1z} u_{1x} & -u_{1z} u_{1y} & u_{1x}^2 + u_{1y}^2 \end{bmatrix} \quad (B19)$$

$$[M_{12}] = \begin{bmatrix} I_{2x} \cos \varphi & 0 & I_{2z} \sin \varphi \\ 0 & 0 & 0 \\ -I_{2x} \sin \varphi & 0 & I_{2z} \cos \varphi \end{bmatrix} \quad (B20)$$

$$[M_{21}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{2y} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (B21)$$

$$[M_{22}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{2y} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (B22)$$

When  $[M_{11}]$ ,  $[M_{12}]$ ,  $[M_{21}]$ , and  $[M_{22}]$  are combined into a  $6 \times 6$  matrix (see eq. (A97)) and the third and fifth rows and columns are deleted, the matrix  $[M]$  is obtained

$$[M] = \begin{bmatrix} I_{1x} + I_{2x} + (I_{2z} - I_{2x})\sin^2\varphi & 0 & (I_{2z} - I_{2x})\sin\varphi \cos\varphi & 0 \\ 0 & I_{1y} & 0 & 0 \\ (I_{2z} - I_{2x})\sin\varphi \cos\varphi & 0 & I_{1z} + I_{2z} - (I_{2z} - I_{2x})\sin^2\varphi & 0 \\ 0 & I_{2y} & 0 & I_{2y} \end{bmatrix} + \frac{M_1 M_2}{M_T} \begin{bmatrix} u_{1x}^2 + u_{1y}^2 & -u_{1x}u_{1y} & -u_{1x}u_{1z} & 0 \\ -u_{1y}u_{1x} & u_{1z}^2 + u_{1x}^2 & -u_{1y}u_{1z} & 0 \\ -u_{1z}u_{1x} & -u_{1z}u_{1y} & u_{1x}^2 + u_{1y}^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (B23)$$

### Calculation of $\{\dot{\Omega}\}$

The first three components of the column vector  $\{\dot{\Omega}\}$  of equation (A99) are the components of the vector  $\dot{\vec{\Omega}}_1$  in the body 1 coordinate system. Because of our linearity assumptions, these components are simply  $\ddot{\theta}_1$ ,  $\ddot{\theta}_2$ , and  $\ddot{\theta}_3$ . The fourth component of  $\{\dot{\Omega}\}$  is the angular acceleration  $\ddot{\varphi}$  of body 2 with respect to body 1. From equation (B6), we can solve for  $\ddot{\varphi}$  in terms of  $\ddot{\theta}_4$ .

$$\ddot{\varphi} = \ddot{\theta}_4 \quad (B24)$$

Thus,  $\{\dot{\Omega}\}$  reduces to

$$\{\dot{\Omega}\} = \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{Bmatrix} \quad (B25)$$

## Calculation of {b}

In calculating the {b} matrix, we neglect all nonlinear terms. The {b} matrix of equation (A99) consists of components of the matrices {b<sub>11</sub>} and {b<sub>22</sub>}. As a first step to calculating {b<sub>11</sub>} and {b<sub>22</sub>} from equations (A96) and (A97), we calculate {q<sub>11</sub>} and {q<sub>22</sub>} from equations (A62) and (A71). Our assumption that forces and torques other than control forces and torques are negligible implies that

$$\{T_{gii}\} = \{F_{sii}\} = \{0\} \quad i = 1, 2$$

So {q<sub>11</sub>} and {q<sub>22</sub>} reduce to

$$\begin{aligned} \{q_{11}\} = & -[I_1]\{ {}_n\dot{\omega}_{11} \} - [J_{\omega 11}][I_1]\{\omega_{11}\} - \frac{M_1 M_2}{M_T} [J_{u1}]\{ {}_nH_{u1} \} \\ & + \{T_{s11}\} + \frac{3\gamma M_1 M_2}{S_m^3 M_T} [J_{u1}][J_{A3}][J_{A3}]\{u_1\} \end{aligned} \quad (B26)$$

$$\{q_{22}\} = -[I_2]\{ {}_n\dot{\omega}_{22} \} - [J_{\omega 22}][I_2]\{\omega_{22}\} + \{T_{s22}\} \quad (B27)$$

We now proceed to evaluate each of the five terms on the right-hand side of equation (B26).

The matrix {<sub>n</sub>ω̇<sub>11</sub>}, from equations (A20), (B1), and (A13), is

$$\{ {}_n\dot{\omega}_{11} \} = [\dot{A}]\{\Omega_{rr}\} = \begin{bmatrix} 0 & \dot{\theta}_3 & -\dot{\theta}_2 \\ -\dot{\theta}_3 & 0 & \dot{\theta}_1 \\ \dot{\theta}_2 & -\dot{\theta}_1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ \omega_o \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\omega_o \dot{\theta}_3 \\ 0 \\ \omega_o \dot{\theta}_1 \end{Bmatrix} \quad (B28)$$

The first term of equation (B26) is then

$$-[I_1]\{ {}_n\dot{\omega}_{11} \} = \begin{Bmatrix} I_{1x} \omega_o \dot{\theta}_3 \\ 0 \\ -I_{1z} \omega_o \dot{\theta}_1 \end{Bmatrix} \quad (B29)$$

The matrix  $\{\omega_{11}\}$ , from equation (A11), is

$$\{\omega_{11}\} = [A]\{\Omega_{rr}\} + \{\Omega_{11}\} = \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \omega_o \\ 0 \end{Bmatrix} + \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} \quad (B30)$$

$$\{\omega_{11}\} = \begin{Bmatrix} \dot{\theta}_1 - \omega_o \theta_3 \\ \dot{\theta}_2 - \omega_o \\ \dot{\theta}_3 + \omega_o \theta_1 \end{Bmatrix} \quad (B31)$$

The second term of equation (B26) is then

$$-[J_{\omega 11}][I_1]\{\omega_{11}\} = \begin{Bmatrix} \omega_{11y}\omega_{11z}(I_{1z} - I_{1y}) \\ \omega_{11z}\omega_{11x}(I_{1x} - I_{1z}) \\ \omega_{11x}\omega_{11y}(I_{1y} - I_{1x}) \end{Bmatrix} = \begin{Bmatrix} -\omega_o(\dot{\theta}_3 + \omega_o \theta_1)(I_{1z} - I_{1y}) \\ 0 \\ -\omega_o(\dot{\theta}_1 + \omega_o \theta_3)(I_{1y} - I_{1x}) \end{Bmatrix} \quad (B32)$$

The matrix  $\{ {}_n H_{u1} \}$ , from equation (A32), (B28), and (B29), is

$$\{ {}_n H_{u1} \} = -[J_{u1}]\{ {}_n \dot{\omega}_{11} \} + [J_{\omega 11}][J_{\omega 11}]\{ u_1 \} \quad (B33)$$

$$\{ {}_n H_{u1} \} = \begin{Bmatrix} -2\omega_o u_2 \dot{\theta}_1 + 2\omega_o u_1 \dot{\theta}_2 + \omega_o^2 u_2 \theta_3 - u_1 \omega_o^2 \\ -\omega_o^2 u_3 \theta_1 + \omega_o^2 u_1 \theta_3 \\ 2\omega_o u_3 \dot{\theta}_2 - 2\omega_o u_2 \dot{\theta}_3 - \omega_o^2 u_2 \theta_1 - u_3 \omega_o^2 \end{Bmatrix} \quad (B34)$$

The third term of equation (B26) is then

$$\begin{aligned}
-\frac{M_1 M_2}{M_T} [J_{u1}] \{ {}_n H_{u1} \} = & -\frac{M_1 M_2}{M_T} \begin{Bmatrix} 2\omega_0 u_3 u_2 \dot{\theta}_2 - 2\omega_0 u_2^2 \dot{\theta}_3 \\ -2\omega_0 u_2 u_3 \dot{\theta}_1 + 2\omega_0 u_2 u_1 \dot{\theta}_3 \\ 2\omega_0 u_2^2 \dot{\theta}_1 - 2\omega_0 u_1 u_2 \dot{\theta}_2 \end{Bmatrix} \\
& -\frac{M_1 M_2}{M_T} \begin{Bmatrix} \omega_0^2 u_3^2 \theta_1 - \omega_0^2 u_2^2 \theta_1 - \omega_0^2 u_1 u_3 \theta_3 - u_3 u_2 \omega_0^2 \\ \omega_0^2 u_1 u_2 \theta_1 + \omega_0^2 u_2 u_3 \theta_3 \\ -\omega_0^2 u_1 u_3 \theta_1 - \omega_0^2 u_2^2 \theta_3 + \omega_0^2 u_1^2 \theta_3 \end{Bmatrix} \quad (B35)
\end{aligned}$$

The fourth term of equation (B26) consists of the control torque of equation (B4).

$$\{ T_{s11} \} = \begin{Bmatrix} -K_1 \theta_1 - C_1 \dot{\theta}_1 \\ -K_2 \theta_2 - C_2 \dot{\theta}_2 \\ -K_3 \theta_3 - C_3 \dot{\theta}_3 \end{Bmatrix} \quad (B36)$$

The matrix  $\{ A_3 \}$  is the third column of  $[A]$ .

$$\{ A_3 \} = \begin{Bmatrix} -\theta_2 \\ \theta_1 \\ 1 \end{Bmatrix} \quad (B37)$$

The fifth term of equation (B26) is then

$$\frac{3\gamma M_1 M_2}{S_m^3 M_T} [J_{u1}] [J_{A3}] [J_{A3}] \{ u_1 \} = \frac{3\gamma M_1 M_2}{S_m^3 M_T} \begin{Bmatrix} u_2^2 \theta_1 - u_3^2 \theta_1 - u_1 u_2 \theta_2 + u_2 u_3 \\ -u_1 u_2 \theta_1 + u_1^2 \theta_2 - u_3^2 \theta_2 - u_1 u_3 \\ u_3 u_1 \theta_1 + u_3 u_2 \theta_2 \end{Bmatrix} \quad (B38)$$

The matrix  $\{ q_{11} \}$  is then given by the sum of equations (B29), (B32), (B35), (B36), and (B38).

To evaluate  $\{ q_{22} \}$  we evaluate each of the three terms on the right-hand side of equation (B27). The matrix  $\{ {}_n \dot{\omega}_{22} \}$ , from equations (A22), (B9), (E1), and (A13), is

$$\{\dot{n}_{22}\} = \begin{Bmatrix} 0 \\ (\omega_0^2 \cos \varphi) \theta_1 + (2\omega_0 \cos \varphi) \dot{\theta}_3 - (\omega_0^2 \sin \varphi) \theta_3 + (2\omega_0 \sin \varphi) \dot{\theta}_1 \\ -(\omega_0^2 \sin \varphi) \theta_1 - (2\omega_0 \sin \varphi) \dot{\theta}_3 - (\omega_0^2 \cos \varphi) \theta_3 + (2\omega_0 \cos \varphi) \dot{\theta}_1 \end{Bmatrix} \quad (B39)$$

The first term of equation (B27) is then

$$-[I_2] \{\dot{n}_{22}\} = \begin{Bmatrix} 0 \\ I_{2y} [ -(\omega_0^2 \cos \varphi) \theta_1 - (2\omega_0 \cos \varphi) \dot{\theta}_3 + (\omega_0^2 \sin \varphi) \theta_3 - (2\omega_0 \sin \varphi) \dot{\theta}_1 ] \\ I_{2z} [ (\omega_0^2 \sin \varphi) \theta_1 + (2\omega_0 \sin \varphi) \dot{\theta}_3 + (\omega_0^2 \cos \varphi) \theta_3 - (2\omega_0 \cos \varphi) \dot{\theta}_1 ] \end{Bmatrix} \quad (B40)$$

Using equations (A16), (B31), (B9), and (B3), the second term of equation (B27) is found to consist entirely of nonlinear terms.

$$-[J_{\omega 22}][I_2]\{\omega_{22}\} = \{0\} \quad (B41)$$

The third term of equation (B27) is also  $\{0\}$ .

$$\{T_{s22}\} = \{0\} \quad (B42)$$

The matrix  $\{q_{22}\}$  is then given by the sum of equations (B40), (B41), and (B42).

Knowing  $\{q_{11}\}$  and  $\{q_{22}\}$  we can now calculate  $\{b_{11}\}$  and  $\{b_{22}\}$  from equations (A96a) and (A96b). Recalling that  $\{T_{FG2}\}$  is the motor torque of equation (B14b),

$$\{T_{FG2}\} = \begin{Bmatrix} 0 \\ (K_4 G \sin \varphi) \theta_1 + K_4 \theta_2 + (K_4 G \cos \varphi) \theta_3 + K_4 \theta_4 + C_4 \dot{\theta}_4 \\ 0 \end{Bmatrix}$$

we can calculate  $\{b\}$  in the matrix form

$$\{b\} = -[C]\{\dot{\theta}\} - [K]\{\theta\} + \{f\} \quad (B43)$$

where  $[C]$ ,  $[K]$ , and  $\{f\}$  are given by



$$[C] = \begin{bmatrix} C_1 + \omega_o^2(I_{2z} - I_{2x})\sin \varphi \cos \varphi & 0 & -\omega_o[I_{1x} - I_{1y} + I_{1z} + 2I_{2x} + 2(I_{2z} - I_{2x})\sin^2 \varphi] & 0 \\ 0 & C_2 & 0 & -C_4 \\ \omega_o[I_{1x} - I_{1y} + I_{1z} + 2I_{2x} + 2(I_{2z} - I_{2x})\cos^2 \varphi] & 0 & C_3 - \omega_o^2(I_{2z} - I_{2x})\sin \varphi \cos \varphi & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \frac{\omega_o M_1 M_2}{M_T} \begin{bmatrix} 0 & 2u_{1y}u_{1z} & -2u_{1y}^2 & 0 \\ -2u_{1y}u_{1z} & 0 & 2u_{1x}u_{1y} & 0 \\ 2u_{1y}^2 & -2u_{1x}u_{1y} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K] = \begin{bmatrix} K_1 - \omega_o^2[I_{1z} - I_{1x} + I_{2x} + (I_{2z} - I_{2x})\sin^2 \varphi] & 0 & -\omega_o^2[(I_{2z} - I_{2x})\sin \varphi \cos \varphi] & 0 \\ -K_4 G \sin \varphi & K_2 - K_4 & -K_4 G \cos \varphi & -K_4 \\ -\omega_o^2[(I_{2z} - I_{2x})\sin \varphi \cos \varphi] & 0 & K_3 - \omega_o^2[I_{1x} - I_{1y} + I_{2x} + (I_{2z} - I_{2x})\cos^2 \varphi] & 0 \\ K_4 G \sin \varphi & K_4 & K_4 G \cos \varphi & K_4 \end{bmatrix}$$

$$+ \frac{\omega_o^2 M_1 M_2}{M_T} \begin{bmatrix} 4(u_{1z}^2 - u_{1y}^2) & 3u_{1x}u_{1y} & -u_{1x}u_{1z} & 0 \\ 4u_{1x}u_{1y} & 3(u_{1z}^2 - u_{1x}^2) & u_{1y}u_{1z} & 0 \\ -4u_{1z}u_{1x} & -3u_{1z}u_{1y} & u_{1x}^2 - u_{1y}^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\{f\} = \frac{\omega_o^2 M_1 M_2}{M_T} \begin{Bmatrix} 4u_{1y}u_{1z} \\ -3u_{1x}u_{1z} \\ u_{1x}u_{1y} \\ 0 \end{Bmatrix}$$

By using equation (B43) for  $\{b\}$ , the final linearized equations of motion can now be written from equation (A99).

$$[M]\{\ddot{\theta}\} + [C]\{\dot{\theta}\} + [K]\{\theta\} = \{f\}$$

## APPENDIX C

### SYMBOLS

$a$	coefficient of $s^3$ in pitch characteristic equation
$[C]$	coefficient matrix of $\{\dot{\theta}\}$
$C_i$	$i^{\text{th}}$ control gain for rate feedback, $i = 1, 2, 3, 4$
$\{f\}$	torque matrix
$G$	parameter determining degree of participation of $\theta_1$ and $\theta_3$ in expression for $\sigma$
$I_x$	effective satellite roll inertia
$I_z$	effective satellite yaw inertia
$I_{1x}, I_{1y}, I_{1z}$	mass moments of inertia of central body about $x, y, z$ axes of central-body coordinate system
$I_{2x}, I_{2y}, I_{2z}$	mass moments of inertia of solar array about $x, y, z$ axes of solar-array coordinate system
$I'_{1y}$	effective central-body pitch inertia
$\Delta I_y$	variation of $I'_{1y}$ from $I_{1y}$
$i$	inclination of equatorial plane to ecliptic plane
$[K]$	coefficient matrix of $\{\theta\}$
$K_i$	$i^{\text{th}}$ control gain for position feedback, $i = 1, 2, 3, 4$
$L$	line formed by projecting earth-sun line into equatorial plane
$[M]$	coefficient matrix of $\{\ddot{\theta}\}$
$M_T$	sum of masses of central body and solar array
$M_1$	mass of central body
$M_2$	mass of solar array
$\text{Re}$	real part of
$s$	Laplace operator
$s_{pi}$	$i^{\text{th}}$ root of pitch characteristic equation, $i = 1, 2, 3, 4$
$s_{ri}$	$i^{\text{th}}$ root of roll characteristic equation, $i = 1, 2$
$s_{yi}$	$i^{\text{th}}$ root of yaw characteristic equation, $i = 1, 2$

$T_{c1}, T_{c2}, T_{c3}$	central-body control torques about x, y, z axes of central-body coordinate system
$T_m$	motor torque on central body
$\bar{u}$	vector from central-body center of mass to solar-array center of mass
$u_{1x}, u_{1y}, u_{1z}$	x, y, z components of vector $\bar{u}$ in central-body coordinate system
$\bar{x}_I, \bar{y}_I, \bar{z}_I$	unit vectors along axes of inertially fixed coordinate system
$\bar{x}_r, \bar{y}_r, \bar{z}_r$	unit vectors along axes of orbital reference coordinate system
$\bar{x}'_r, \bar{y}'_r, \bar{z}'_r$	unit vectors along axes of orbital reference coordinate system with origin translated to center of earth
$\bar{x}_1, \bar{y}_1, \bar{z}_1$	unit vectors along axes of central-body coordinate system
$\bar{x}'_1, \bar{y}'_1, \bar{z}'_1$	unit vectors along axes of central-body coordinate system with origin translated to solar-array center of mass
$\bar{x}_2, \bar{y}_2, \bar{z}_2$	unit vectors along axes of solar-array coordinate system
$[Z]$	impedance matrix
$z_{ij}$	element in $i^{\text{th}}$ row, $j^{\text{th}}$ column of $[Z]$
$\alpha$	angle subtended by line L and vector $z'_r$ evaluated at time equal to zero
$\gamma$	earth gravitational constant
$\bar{\delta}$	unit vector directed from solar-array center of mass to sun
$\bar{\delta}_p$	unit vector formed by projecting $\bar{\delta}$ into equatorial plane
$\theta_i$	$i^{\text{th}}$ angular coordinate, $i = 1, 2, 3, 4$
$\theta_s$	apparent angular displacement of sun from autumnal equinox measured in ecliptic plane
$\{\theta\}$	column matrix of angular coordinates
$\{\dot{\theta}\}$	column matrix of angular rates
$\{\ddot{\theta}\}$	column matrix of angular accelerations
$\{\theta_{ss}\}$	column matrix of steady-state displacements
$\sigma$	solar-array error
$\varphi$	angular rotation of solar-array coordinate system with respect to central-body coordinate system
$\omega_o$	angular velocity of circular synchronous orbit

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4. Grote, Philip B.; McMunn, John C.; and Gluck, Rafael: Equations of Motion of Flexible Spacecraft. J. Spacecraft Rockets, vol. 8, no. 6, June 1971, pp. 561-567.

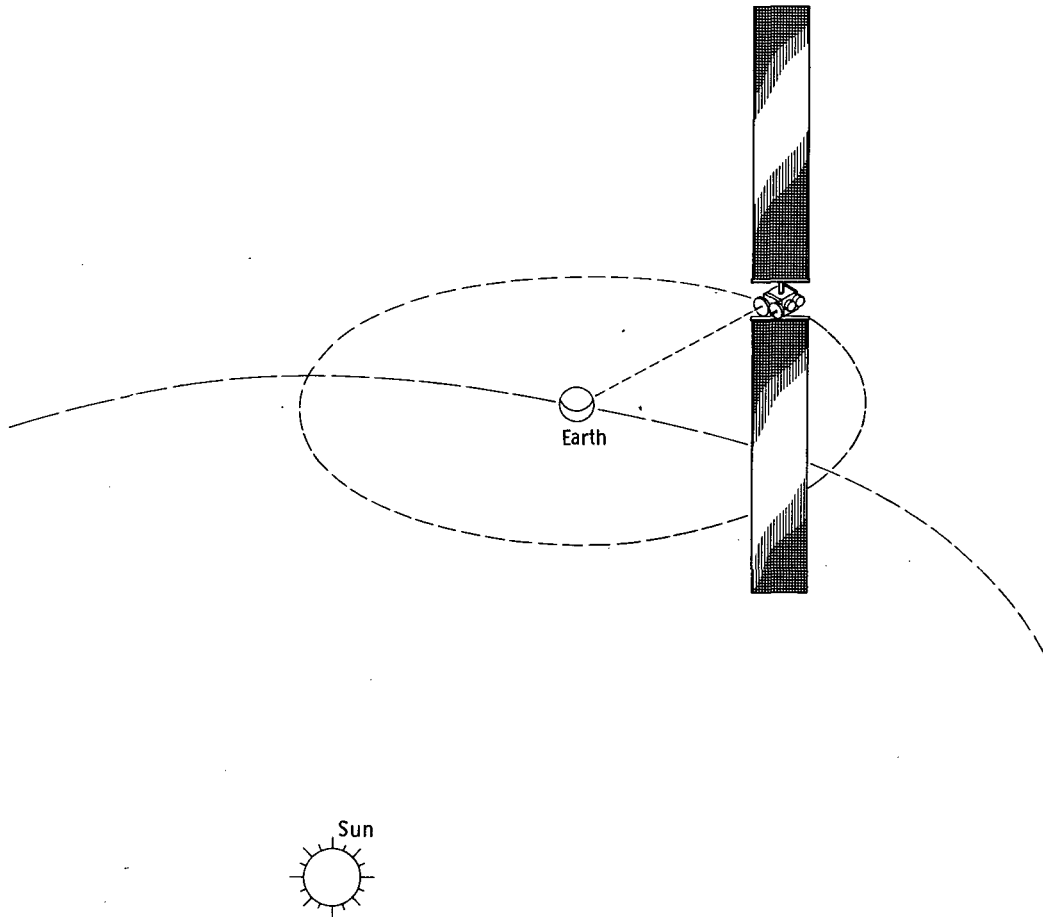


Figure 1. - Configuration of high-power communication satellite.

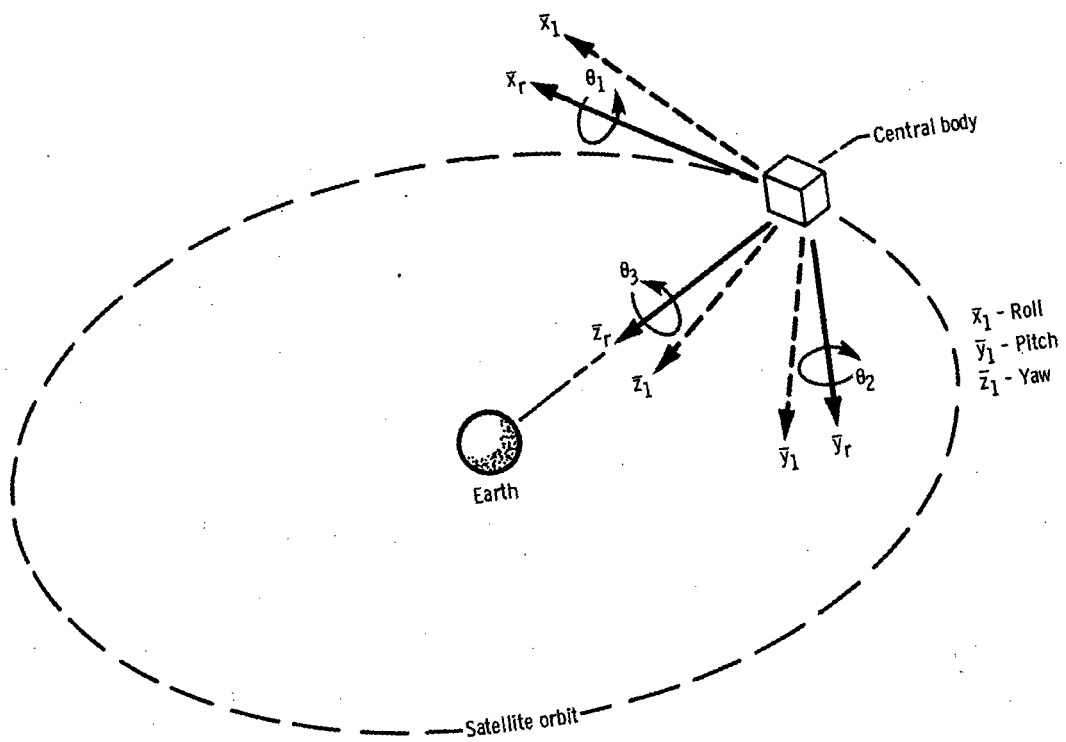


Figure 2. - Orbital reference coordinate system and central-body coordinate system.

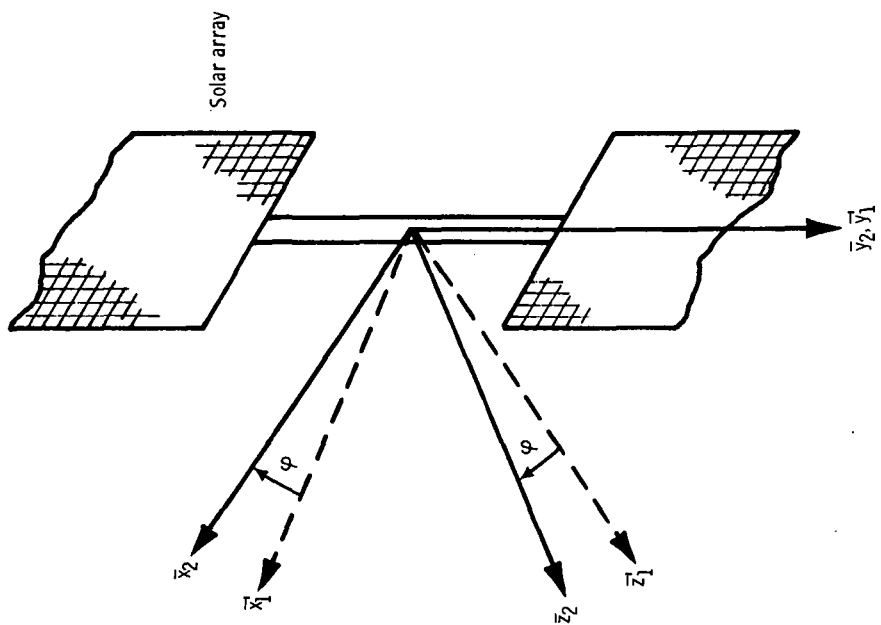


Figure 3. - Solar-array coordinate system.

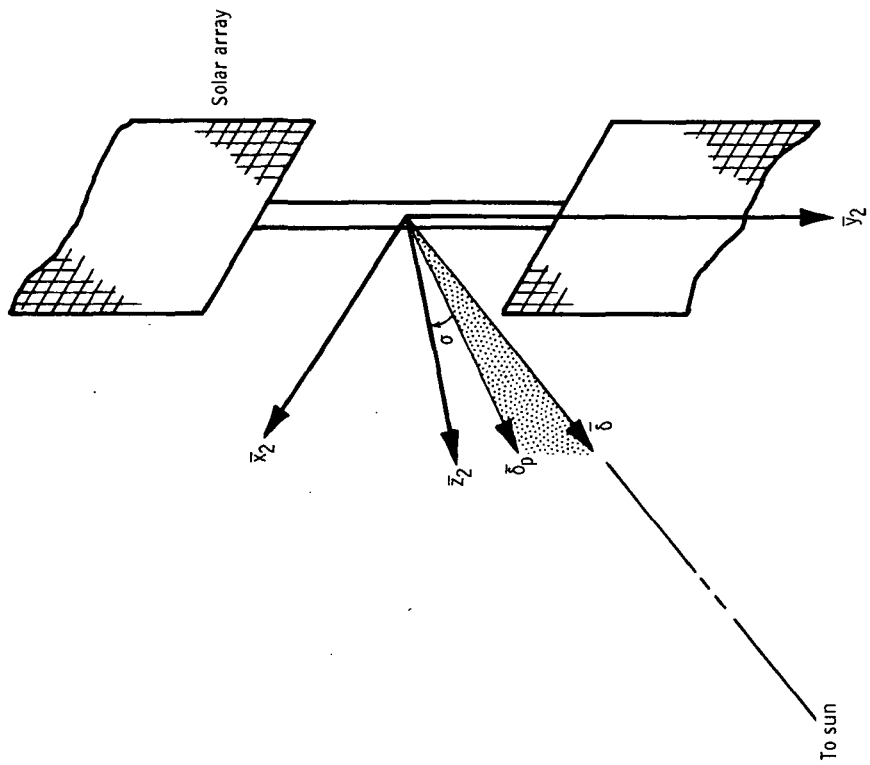


Figure 4. - Solar-array pointing error  $\sigma$ .

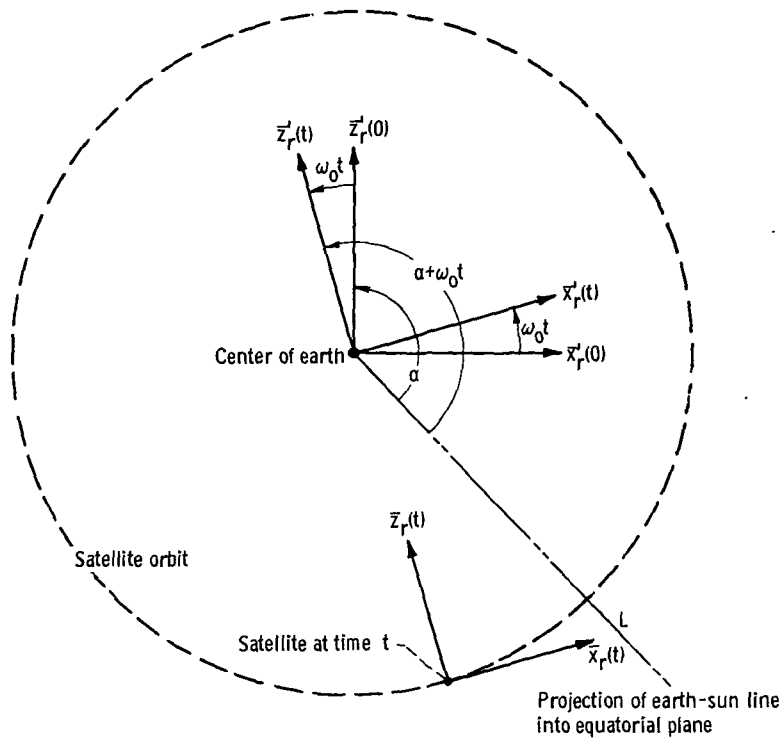


Figure 5. - Geometry for defining  $\theta_4$ .

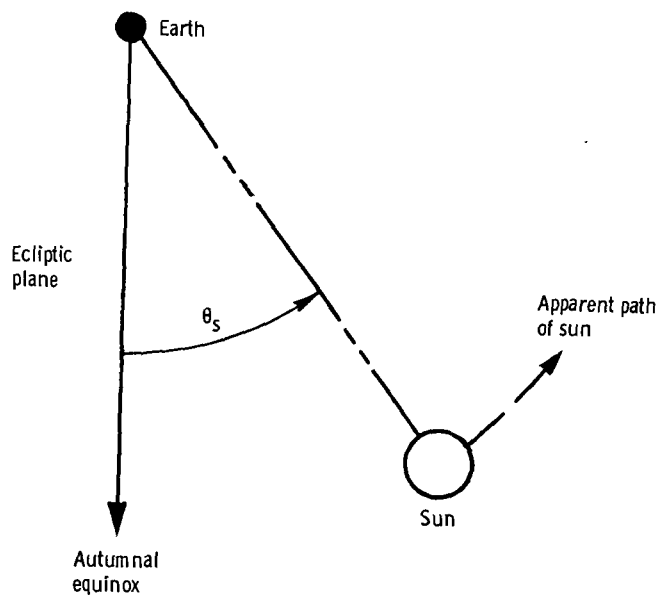


Figure 6. - Sun angle  $\theta_s$  measured in ecliptic plane.



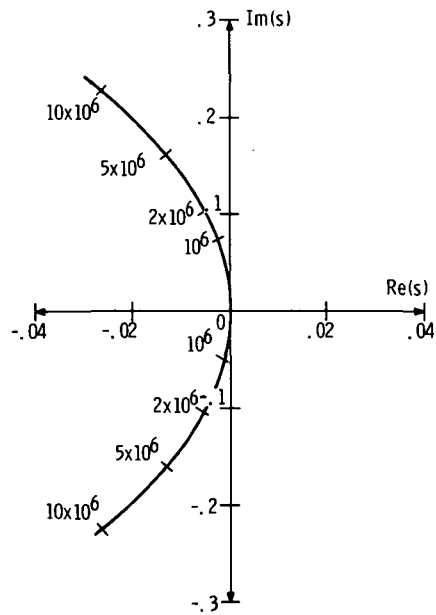


Figure 7. - Locus of roll-yaw roots for coincident centers of mass. Small rate feedback; locus parameter,  $K_1/\omega_0^2 (I_{1x} + I_{2x})$ .

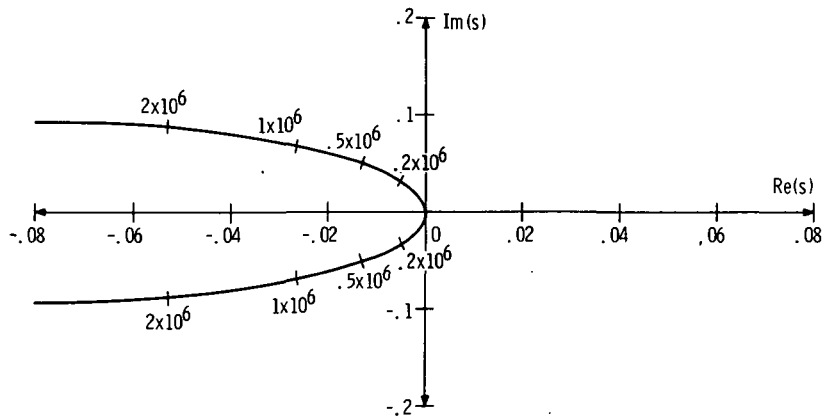


Figure 8. - Locus of roll-yaw roots for coincident centers of mass. Large rate feedback; locus parameter,  $K_1/\omega_0^2 (I_{1x} + I_{2x})$ .

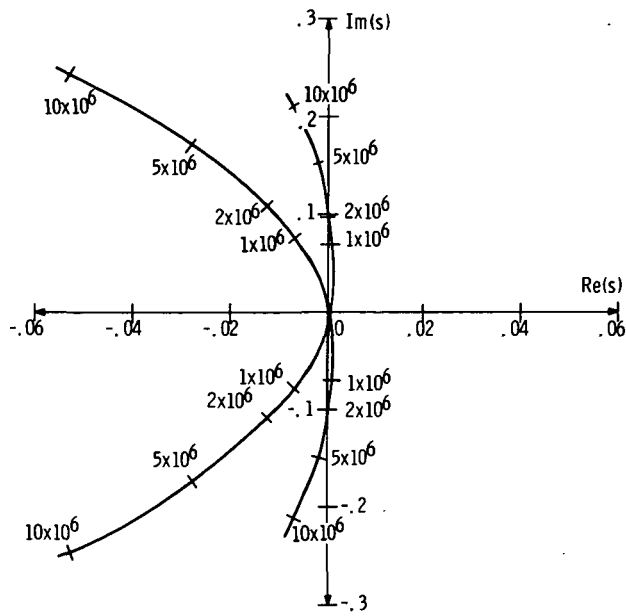


Figure 9. - Locus of pitch roots for coincident centers of mass. Small rate feedback; locus parameter,  $K_4/\omega_0^2 I_{2y}$ .

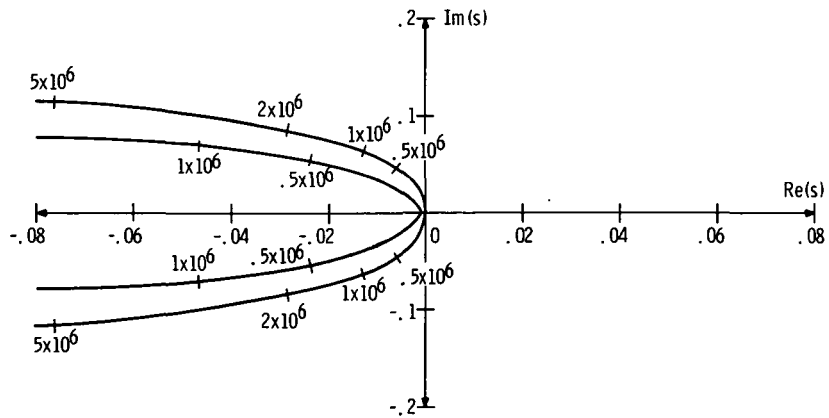


Figure 10. - Locus of pitch roots for coincident centers of mass. Large rate feedback; locus parameter,  $K_4/\omega_0^2 I_{2y}$ .

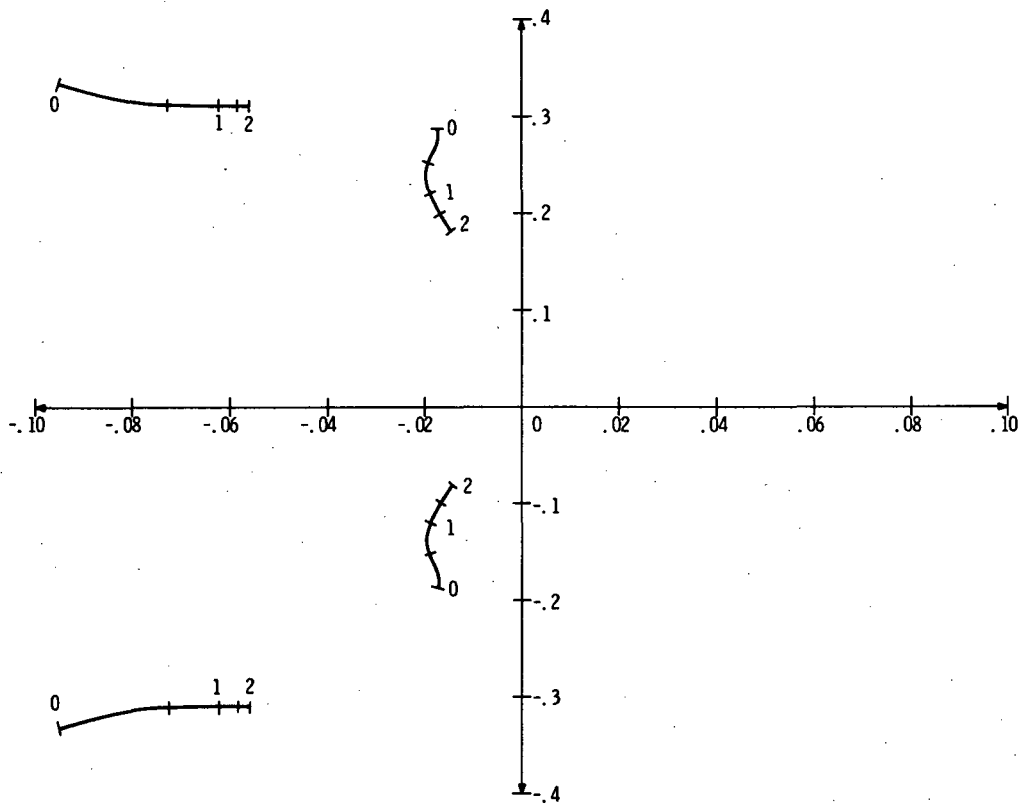


Figure 11. - Locus of pitch roots for distinct centers of mass. Locus parameter,  $\Delta I_{ly}/I_{ly}$ .

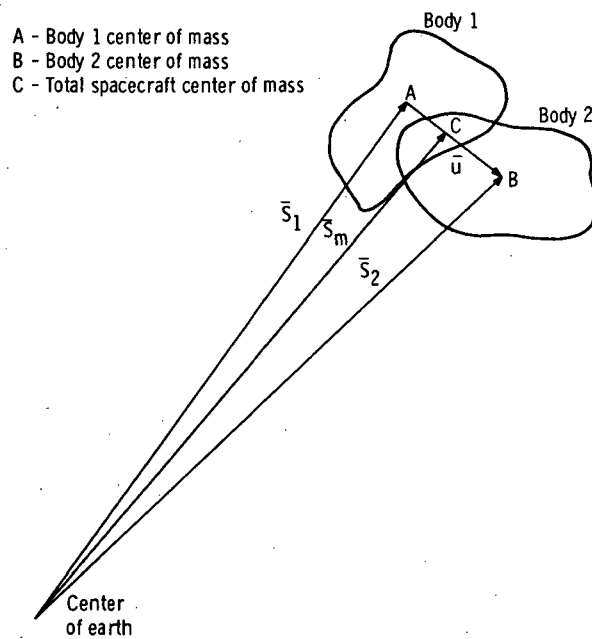


Figure 12. - Illustration of vectors  $\bar{u}$ ,  $\bar{S}_1$ ,  $\bar{S}_2$ ,  $\bar{S}_m$ .

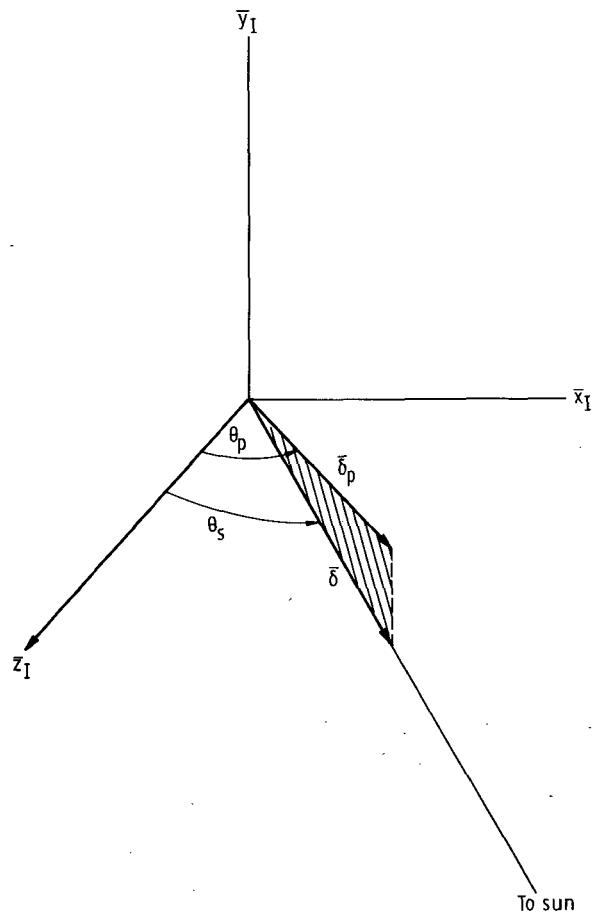


Figure 13. - Illustration of angles  $\theta_p$  and  $\theta_s$  and vectors  $\bar{\delta}$  and  $\bar{\delta}_p$ .



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— NATIONAL AERONAUTICS AND SPACE ACT OF 1958

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